# Tetrad Gravity: III) Asymptotic Poincaré Charges, the Physical Hamiltonian and Void Spacetimes.

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## Abstract

After a review on asymptotic flatness, a general discussion of asymptotic weak and strong Poincaré charges in metric gravity is given with special emphasis on the boundary conditions needed to define the proper Hamiltonian gauge transformations and to get a differentiable Dirac Hamiltonian. Lapse and shift functions are parametrized in a way which allows to identify their asymptotic parts with the lapse and shift functions of Minkowski spacelike hyperplanes. After having added the strong (surface integrals) Poincaré charges to the Dirac Hamiltonian, it becomes the sum of a differentiable Hamiltonian and of the weak (volume integrals) Poincaré charges. By adding the ten Dirac extra variables at spatial infinity, which identify special families of foliations with leaves asymptotic (in a direction-independent way) to Minkowski spacelike hyperplanes, metric gravity is extended to englobe Dirac's ten extra first class constraints which identify the weak Poincaré charges with the momenta conjugate to the extra variables. This opens the path to a consistent deparametrization of general relativity to parametrized Minkowski theories restricted to spacelike hyperplanes. The requirement of absence of supertranslations restricts: i) the boundary conditions on the fields and the gauge transformations to those identifying the family of Christodoulou-Klainermann spacetimes; ii) the allowed 3+1 splittings of spacetime to those whose spacelike leaves correspond to the Wigner hyperplanes of Minkowski parametrized theories [on these leaves, named Wigner-Sen-Witten hypersurfaces, there is

a rule of parallel transport determined by the Sen-Witten connection]. This approach is extended to tetrad gravity with its interpretation of the super-hamiltonian constraint as a generator of gauge transformations: the last gauge variable is the momentum $\rho(\tau,\vec{\sigma})$ conjugate to the conformal factor $q(\tau,\vec{\sigma})$ of the 3-metric ($q$ has to be determined as a solution of the constraint, namely of the Lichnerowicz equation). In the 3-orthogonal gauges, the further addition of the natural gauge fixing $\rho(\tau,\vec{\sigma}) \approx 0$ leads to a reduced phase space, which is parametrized by the canonical variables $r_{\bar{a}}(\tau,\vec{\sigma})$, $\pi_{\bar{a}}(\tau,\vec{\sigma})$, $\bar{a} = 1, 2$, defining the Hamiltonian kinematical gravitational field. The evolution in the parameter labelling the leaves of the foliation is generated by the reduced ADM energy in the rest frame. Then, "void spacetimes", gauge equivalent to Minkowski spacetime in rectangular coordinates, are defined and their realization in the 3-orthogonal gauges is given by adding by hand the two pairs of second class constraints $r_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$, $\pi_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$, which are compatible with Einstein's equations in absence of matter. These void spacetimes are the maximal extension of the non-inertial Galilean reference frames of Newtonian gravity to Einstein general relativity. Some comments on the quantization of the theory are done.

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## I. INTRODUCTION

In the first two papers [1,2], quoted as I and II respectively, a new formulation of tetrad gravity was given and its Dirac observables were found in 3-orthogonal coordinates on the Cauchy hypersurfaces $\Sigma_\tau$ (assumed diffeomorphic to $R^3$) of the 3+1 splitting of a globally hyperbolic spacetime $M^4$, asymptotically flat at spatial infinity. This description is assumed valid, in a variational sense, for an interval $\triangle\tau$ of the time parameter $\tau$ labelling the leaves of the foliation associated with the 3+1 splitting, after which conjugate points of the 3-geometry on $\Sigma_\tau$ and/or 4-dimensional singularities develop due to Einstein's equations.

In II there was a preliminary discussion of the Hamiltonian group of gauge transformations, whose generators are the 14 first class constraints of the model. Also some preliminary statement about the parametrization of the lapse and shift functions at spatial infinity was made. However, only some indications on the boundary conditions for the fields of tetrad gravity were given and there was no statement about the boundary conditions on gauge transformations needed to define proper gauge transformations; only some general remarks on the necessity of using weighted Sobolev spaces to avoid isometries and Gribov ambiguities were done.

In the conclusions of II it was emphasized the difference between the Hamiltonian group $\bar{\mathcal{G}}$ of gauge transformations and the group of spacetime diffeomorphisms $Diff\, M^4$ in connection with the definition of observables. There was the distinction between "Hamiltonian kinematical gravitational fields" (equivalence classes of spacetimes modulo the group of Hamiltonian gauge transformations: in general every equivalence class contains many gauge-equivalent 4-geometries, namely many standard "kinematical gravitational fields", elements of $Riem\, M^4/Diff\, M^4$) and "Hamiltonian Einstein or dynamical gravitational fields", which are those kinematical ones which also satisfy the Hamilton-Dirac equations. It was shown that the Hamiltonian dynamical gravitational fields coincide with the standard "Einstein or dynamical gravitational fields", namely with a single 4-geometry whose 4-metrics are solution of Einstein's equations; this 4-geometry is parametrized in terms of one conformal 3-geometry. Therefore, on the space of solutions of Einstein's equations the spacetime diffeomorphisms in $Diff\, M^4$ become dynamical symmetries of Einstein's equations satisfying their associated Jacobi equations (or linearized Einstein equations) and contain as a subset the allowed Hamiltonian gauge transformations, namely those gauge transformations which are dynamical symmetries of the Hamilton-Dirac equations. There was also reported the criticism of Bergmann [3] about general covariance.

In this paper, after having identified the problems (like the existence of supertranslations among the asymptotic symmetries) present in spacetimes asymptotically flat at spatial infinity, we study the differentiability of the Dirac Hamiltonian, how to define proper Hamiltonian gauge transformations and the asymptotic conserved weak and strong Poincaré charges (the analogue of the non Abelian charges of Yang-Mills theory) in metric gravity, and then in tetrad gravity, following the ADM linearized theory [4], Regge-Teitelboim [5]and Beig-O'Murchadha [6].

We shall see that the final set of boundary conditions on the fields and the gauge transformations, which allow the elimination of supertranslations, the preservation of this property under the allowed gauge transformations and a good definition of $\Sigma_\tau$-adapted tetrads at spatial infinity, identify the family of spacetimes defined by Christodoulou and Klainermann

[7] as the one which admits a standard Hamiltonian formulation. All the quantities must have a direction-independent limit at spatial infinity like in Yang-Mills theory. In particular, following the approach of Dirac [8,9] to add 10 degrees of freedom at spatial infinity and 10 extra first class constraints to make them gauge variables, we get that the allowed 3+1 splittings of $M^4$ must be foliations whose spacelike leaves $\Sigma_\tau$ tend asymptotically to Minkowski spacelike hyperplanes in a direction-independent way. The requirement of absence of supertranslations further restricts the foliations to those (named Wigner-Sen-Witten) with the leaves $\Sigma_\tau$ asymptotically normal to the ADM 4-momentum (definition of rest frame). On these hypersurfaces there is a rule of parallel transport determined by the Sen-Witten connection. The asymptotic Minkowskian Wigner hyperplanes [10] identify "geometrical and dynamical" preferred observers at spatial infinity and there is a realization of Bergmann's expectation that it should be possible to restrict proper coordinate transformations to contain an invariant Poincaré subgroup plus asymptotically trivial diffeomorphisms. The evolution in the parameter labelling the leaves of the foliation and identified with the rest-frame time of a decoupled observer (point clock) sitting in the "external center of mass" of the universe, is governed by the ADM energy.

There is a justification why the superhamiltonian constraint has to be solved in the conformal factor of the 3-metric, becoming the reduced Lichnerowicz equation, so that the last gauge variable of tetrad gravity is its conjugate momentum. Since the ADM theory is independent from the choice of the 3+1 splitting of spacetime, it turns out that the transition from an allowed 3+1 splitting to another one is realized with a gauge transformation generated by the superhamiltonian constraint.

Then, there is the definition of the equivalence class of "void spacetimes", namely those spacetimes in which the physical degrees of freedom of the gravitational field have been explicitly eliminated. These spacetimes are gauge equivalent to Minkowski spacetime in rectangular coordinates due to the Hamiltonian group of gauge transformations. They turn out to have the spacelike Cauchy surfaces 3-conformally flat, to have vanishing Poincaré charges and to require a different Hamiltonian treatment without adding Dirac's extra variables. In accord with the fact that in parametrized Minkowski theories Wigner hyperplanes may be defined only in presence of matter, void spacetimes do not allow the definition of Wigner-Sen-Witten hypersurfaces.

This concludes the study of the Hamiltonian formulation and canonical reduction of classical tetrad gravity in absence of matter. All its conceptual problems have found an explanation and there is an indication on how to quantize the theory in terms of Dirac's observables after a complete breaking of general covariance. The study of the theory in presence of matter and its linearization in the 3-orthogonal gauge will be done in future papers. The main obstacle in this standard Hamiltonian treatment is the lack of solutions of the Lichnerowicz equation: this explains why research has shifted towards either Ashtekar's programme or superstring theory. We hope that after this clarification of the conceptual aspects of tetrad gravity and after having found the bridge both to parametrized Minkowski and Newtonian theories, the search of exact and/or approximate solutions of the Lichnerowicz equation will start again, also because our results imply a unification of the techniques between Yang-Mills theory and tetrad gravity. Moreover, the problems with the gauge fixings in numerical gravity are deeply connected with a better understanding of these solutions.

In Section II there is a review of the properties of spacetimes asymptotically flat at either

null or spatial infinity in metric gravity.

In Section III, after a review of Poincaré charges and of the problem of supertranslations in metric gravity, we give the boundary conditions on the 3-metric and the parametrization of the lapse and shift functions allowing to get a differentiable Dirac Hamiltonian and to define proper gauge transformations. Weak and strong Poincaré charges emerge naturally. Two scenarios for metric gravity are delineated, one without and the other with Dirac's ten extra degrees of freedom at spatial infinity. In the second case, ten extra first class constraints are added so that the ten extra degrees of freedom are eliminable gauge variables. Therefore, one has the same physical degrees of freedom of the gravitational field and, moreover, one can make contact with parametrized Minkowski theories on spacelike hyperplanes, which are the natural limit for $G \rightarrow 0$ when matter is present.

In Section IV it is shown that the analogue of Minkowski spacelike hyperplanes are hypersurfaces, tending to them at spatial infinity in a direction-independent way, and having a rule of parallel transport, which is determined by the Sen-Witten connection for those hypersurfaces (named Wigner-Sen-Witten hypersurfaces) tending to Wigner hyperplanes in Minkowski spacetime in absence of gravity.

In Section V the previous results are extended to the formulation of tetrad gravity given in I and II, and, then, are specialized to the 3-orthogonal gauges. After the introduction of a natural gauge fixing for the superhamiltonian constraint, the canonical degrees of freedom of the gravitational field in this special gauge are found and it is shown that the ADM energy plays the role of the physical Hamiltonian for the evolution in the parameter labelling the leaves of the foliation defining the 3+1 splitting of spacetime.

In Section VI we define "void spacetimes" as those special spacetimes in which there are no canonical degrees of freedom of the gravitational field. They are gauge equivalent to Minkowski spacetime in rectangular coordinates by using the Hamiltonian gauge transformation group according to constraint theory. They contain the analogue of the inertial forces of Newtonian gravity appearing when non-orthogonal (maybe time-dependent) coordinates are used for Euclidean space (Galilean non-inertial reference frames).

In the Conclusions there is a summary of the main results, the description of our way out from the "problem of time", and some remarks on how to quantize (at least formally) tetrad gravity in the special 3-orthogonal gauge, after a complete breaking of general covariance, without problems with the definition of the physical scalar product and with the implication that volumes, areas and lengths on the Cauchy surfaces of the mathematical spacetime (in this completely fixed gauge) are going to be quantized (even if in a way different from Ashtekar's one). There is also a comment on Komar-Bergmann's proposal of identifying a posteriori the "physical points" of spacetime with their "individuating fields".

In Appendix A there is the application of the second Noether theorem to ADM metric gravity and the explicit check that the ADM action is not invariant under spacetime diffeomorphisms.

In Appendix B there is a review of proposals for the reduced phase space of metric gravity.

In Appendix C there is a review of spinors on $M^4$ and $\Sigma_\tau$.

In Appendix D, for the sake of completeness, there is the application of the second Noether theorem to the Einstein-Hilbert action, the definition of Komar superpotential and of the energy-momentum pseudotensors of metric gravity.

In Appendix E there is the expression of 3- and 4-tensors in void spacetimes.

In Appendix F there is the definition of a set of null tetrads natural for the Hamiltonian formulation of tetrad gravity.

In Appendix G there is the connection with the post-Newtonian approximation.

## II. ASYMPTOTIC FLATNESS AT SPATIAL INFINITY.

The definition of an "isolated system" in general relativity [see Ref. [11] for a review] is a difficult problem, since there is neither a background flat metric ${}^4\eta$ nor a natural global inertial coordinate system allowing to define a preferred radial coordinate $r$ and a limit ${}^4g_{\mu\nu} \rightarrow {}^4\eta_{\mu\nu} + O(1/r)$ for $r \rightarrow \infty$ along either spatial or null directions. Usually, one considers an asymptotic Minkowski metric ${}^4\eta_{\mu\nu}$ in rectangular coordinates [${}^4\eta_{\mu\nu} = \epsilon(+---)$, with $\epsilon = +1$ for the particle physics convention and $\epsilon = -1$ for the general relativity one] and tries to get asymptotic statements with various types of definitions of $r$. However, it is difficult to correctly specify the limits for $r \rightarrow \infty$ in a meaningful, coordinate independent way.

Therefore, Penrose [12] introduced the notions of asymptotic flatness at null infinity (i.e. along null geodesics) and of asymptotic simplicity with his conformal completion approach [a spacetime $(M^4, {}^4g)$ is asymptotically simple if: i) all its null geodesics are complete; ii) there exists a smooth function $\Omega \geq 0$ going to zero in both directions along null geodesics; iii) there exists a smooth extension $\tilde{M}^4 = M^4 \cup \mathcal{I}$ of $M^4$ with boundary $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$, $\mathcal{I}^+ \cap \mathcal{I}^- = 0$; iv) $\Omega$ extends smoothly to $\mathcal{I}$, where $\Omega = 0$, $d\Omega \neq 0$; v) ${}^4\tilde{g} = \Omega^2\, {}^4g$ is a smooth 4-metric on $\tilde{M}^4$; vi) each null geodesic acquires an endpoint in the past on $\mathcal{I}^-$ and in the future on $\mathcal{I}^+$]. See chapter 11 of Ref. [13] for the conformal completion of Minkowski spacetime [through a conformal isometry it becomes an open region (with boundary $\Omega = 0$) of the Einstein static spacetime $S^3 \times R$; the boundary contains two points $i^{\pm}$ for the future and past timelike infinity, one point $i^o$ for spacelike infinity, and two regions $\mathcal{I}^{\pm}$ for future and past null infinity] and for a review of asymptotic flatness. See also Ref. [14] for definitions of "asymptotically simple and weakly asymptotically simple" spacetimes, intended to ensure that the asymptotic structure be globally the same as that of Minkowski spacetime.

Instead, a coordinate-independent definition of asymptotic flatness at spatial infinity in terms of the "large distance" behaviour of initial data on a Cauchy surface was introduced by Geroch [15].

Then, the two definitions of asymptotic flatness at null and spatial infinity were unified in the SPI formalism of Ashtekar and Hanson [16] (see the bibliography of this reference for other approaches like the projective one [17], which, however, has problems with Schwarzschild spacetime). Essentially, in the SPI approach, the spatial infinity of the spacetime $M^4$ is compactified to a point $i^o$ [instead in the projective approach of Ref. [17] a timelike hyperboloid is the boundary of $M^4$] and fields on $M^4$ have direction-dependent limits at $i^o$ [this state of affairs implies a peculiar differential structure on $\Sigma_\tau$ and awkward differentiability conditions of the 4-metric]. Subsequently, in Ref. [18], a new kind of completion, neither conformal nor projective, is developed by Ashtekar and Romano: now the boundary of $M^4$ is a unit timelike hyperboloid like in the projective approach, which, however, has a well defined contravariant normal in the completion [different conformal rescalings of the 4-metric ${}^4g \mapsto {}^4\tilde{g} = \Omega^2\, {}^4g$ ($\Omega \geq 0$, $\Omega = 0$ is the boundary 3-surface of the unphysical spacetime $\tilde{M}^4$) and of the normal $n^\mu \mapsto \tilde{n}^\mu = \Omega^{-4}\, n^\mu$ are needed]; now, there is no need of awkward differentiability conditions. While in the SPI framework each hypersurface $\Sigma_\tau$ has the sphere at spatial infinity compactified at the same point $i^o$, which is the vertex for both future $\mathcal{I}^+$ (scri-plus) and past $\mathcal{I}^-$ (scri-minus) null infinity, these properties are lost in the new approach: each $\Sigma_\tau$ has as boundary at spatial infinity the sphere cut by $\Sigma_\tau$ in the

timelike hyperboliod; there is no relation between the timelike hyperboloid at spatial infinity and $\mathcal{I}^{\pm}$. This new approach simplifies the analysis of Ref. [19] of uniqueness (modulo the logarithmic translations of Bergmann [20]) of the completion at spacelike infinity.

There is a coordinate-dependent formalism of Beig and Schmidt [21] [developed to avoid the awkward differentiability conditions of the SPI framework and using polar coordinates like the standard hyperbolic ones for Minkowski spacetime and agreeing with them at "first order in 1/r"], whose relation to the new completion is roughly the same as that between Penrose's coordinate-independent approach to null infinity [12] and Bondi's approach [22] based on null coordinates. The class of spacetimes studied in Ref. [21] [called "radially smooth of order m" at spatial infinity] have 4-metrics of the type

$$ds^2 = d\rho^2 (1 + \frac{{}^1\sigma}{\rho} + \frac{{}^2\sigma}{\rho^2} + ..)^2 + \rho^2 ({}^o h_{rs} + \frac{1}{\rho}\, {}^1 h_{rs} + ..) d\phi^r d\phi^s,$$

where ${}^o h_{rs}$ is the 3-metric on the unit timelike hyperboloid and ${}^n\sigma$, ${}^n h_{rs}$, are functions on it. There are coordinate charts $x^\sigma$ in $(M^4, {}^4g)$ where the 4-metric becomes

$${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + \sum_{n=1}^{m} \frac{1}{\rho^n}\, {}^n l_{\mu\nu}(\frac{x^\sigma}{\rho}) + O(\rho^{-(m+1)}).$$

See Ref. [23] for the status of the conformal field equations, derived from Einstein's equations, which arise in the study of the compatibility of Penrose's conformal completion approach with Einstein's equations; the final outcome is a description in which $i^o$ blows up to a 2-sphere at spatial infinity (interpretable as the space of spacelike directions at $i^o$), which intersects future $\mathcal{I}^+$ and past $\mathcal{I}^-$ null infinity in two sets $I^{\pm}$. It is an open question whether the concepts of asymptotic simplicity and conformal completion are too strong requirements. For instance, the Christodoulou and Klainerman [7] result on the nonlinear gravitational stability of Minkowski spacetime implies a peeling behaviour of the conformal Weyl tensor near null infinity which is weaker than the peeling behaviour implied by asymptotic simplicity [see Ref. [22,12]] and this could mean that asymptotic simplicity can be established only, if at all, with conditions stronger than those required by these authors. In Ref. [7] one studies the existence of global, smooth, nontrivial solutions to Einstein's equations without matter, which look, in the large, like the Minkowski spacetime (without singularities and black holes; since the requirements needed for the existence of a conformal completion are not satisfied, it is possible to evade the singularity theorems), are close to Minkowski spacetime in all directions in a precise manner (for developments of the initial data sets uniformly close to the trivial one) and admit gravitational radiation in the Bondi sense. These author's reformulate Einstein's equations with the ADM variables (there are four constraint equations plus the equations for $\partial_\tau\, {}^3g_{rs}$ and $\partial_\tau\, {}^3K_{rs}$; see Section V of I for their phase space counterpart), put the shift functions equal to zero (the lapse function is assumed equal to 1 at spatial infinity, but not everywhere because, otherwise, one should have a finite time breakdown) and add the maximal slicing condition ${}^3K = 0$. Then, they assume the existence of a coordinate system $\vec{\sigma}$ near spatial infinity on the Cauchy surfaces $\Sigma_\tau$ and of smoothness properties for ${}^3g_{rs}$, ${}^3K_{rs}$, such that for $r = \sqrt{\vec{\sigma}^2} \rightarrow \infty$ the initial data set $(\Sigma_\tau, {}^3g_{rs}, {}^3K_{rs})$ is "strongly asymptotically flat", namely [$f(\vec{\sigma})$ is $o_m(r^{-k})$ if $\partial^l f = o(r^{-k-l})$ for $l = 0, 1, .., m$ and $r \rightarrow \infty$]

$$
\begin{aligned}
{}^3g_{rs} &= (1+\frac{M}{r})\delta_{rs} + o_4(r^{-3/2}),\\
{}^3K_{rs} &= o_3(r^{-5/2}),
\end{aligned}
\tag{1}
$$

where the leading term in ${}^3g_{rs}$ is called the Schwarzschild part of the 3-metric, also in absence of matter; this asymptotic behaviour ensures the existence of the ADM energy and angular momentum and the vanishing of the ADM momentum ("center-of-mass frame"). The addition of a technical global smallness assumption on the strongly asymptotically flat initial data leads to a unique, globally hyperbolic, smooth and geodesically complete solution of Einstein's equations without matter, which is globally asymptotically flat in the sense that its Riemann curvature tensor approaches zero on any causal or spacelike geodesic. It is also shown that the 2-dimensional space of the dynamical degrees of freedom of the gravitational field at a point (the reduced configuration space, see the Conclusions of II) is the space of trace-free symmetric 2-covariant tensors on a 2-plane. A serious technical difficulty (requiring the definition of an 'optical function' and reflecting the presence of gravitational radiation in any nontrivial perturbation of Minkowski spacetime) derives from the 'mass term' in the asymptotic Schwartzschild part of the 3-metric: it has the long range effect of changing the asymptotic position of the null geodesic cone relative to the maximal (${}^3K = 0$) foliation (these cones are expected to diverge logarithmically from their positions in flat spacetime and to have their asymptotic shear drastically different from that in Minkowski spacetime).

Other reviews of the "problem of consistency", i.e. whether the geometric assumptions inherent in the existing definitions of asymptotic flatness are compatible with Einstein equations, are given in Refs. [11,24], while in Ref. [25] a review is given about spacetimes with gravitational radiation (nearly all the results on radiative spacetimes are at null infinity, where, for instance, the SPI requirement of vanishing of the pseudomagnetic part of the Weyl tensor to avoid supertranslations is too strong and destroys radiation). See Ref. [26] for a recent review of the state of the art.

Let us now consider the problem of asymptotic symmetries [13] and of the associated conserved asymptotic charges containing the ADM Poincaré charges (which will be discussed in the next Section).

Like null infinity admits an infinite-dimensional group (the BMS group [22]) of "asymptotic symmetries", the SPI formalism admits an even bigger group, the SPI group [16], of such symmetries. Both BMS and SPI algebras have an invariant 4-dimensional subalgebra of translations, but they also have invariant infinite-dimensional Abelian subalgebras (including the translation subalgebra) of so called "supertranslations" [angle (or direction)-dependent translations]. Therefore, there is an infinite number of copies of Poincaré subalgebras in both BMS and SPI algebras, whose Lorentz parts are conjugate through supertranslations [the quotient of BMS and SPI groups with respect to supertranslations is isomorphic to a Lorentz group]. All this implies that there is no unique definition of Lorentz generators and that in general relativity one cannot define intrinsically angular momentum and the Poincaré spin Casimir, so important for the classification of particles in Minkowski spacetime. In Ref. [27] it is shown that the only known Casimirs of the BMS group are $p^2$ and one its generalization involving supertranslations. While Poincaré asymptotic symmetries correspond to the ten Killing fields of the Minkowski spacetime (to which an asymptotically flat spacetime tends asymptotically in some way depending on the chosen definition of asymptotic flatness), supertranslations are "angle-dependent translations", which come just as close to satisfying

Killing's equations asymptotically as any Poincaré transformation [13]. The problem seems to be that all known function spaces, used for the 4-metric and for Klein-Gordon and electromagnetic fields, do not put any restriction on the asymptotic angular behaviour of the fields, but only restrict their radial decrease. Due to the relevance of the Poincaré group for particle physics in Minkowski spacetime, and also to have a good definition of angular momentum in general relativity [see Refs. [28,13,16] for this topic], one usually restricts the class of spacetimes with boundary conditions such that supertranslations are not allowed to exist. In the SPI framework [16], one asks that the pseudomagnetic part of the limit of the conformally rescaled Weyl tensor vanishes at $i^o$. In Ref. [29] a 3+1 decomposition is made of the SPI framework; after having reexpressed the conserved quantities at $i^o$ in terms of canonical initial data, it is shown that to remove ambiguities connected with the supertranslations one must use stronger boundary conditions again implying the vanishing of the pseudomagnetic part of the Weyl tensor.

A related approach to these problems is given by Anderson in Ref. [30]. He proved a slice theorem for the action of spacetime diffeomorphisms asymptotic to Poincaré transformations on the set of asymptotically flat solutions of Einstein's equations in the context of spatial infinity, maximal slicing and asymptotic harmonic coordinates (as gauge conditions). There is a heuristic extension of the method of reduction of dynamical systems with symmetries to the diffeomorphism group. The standard method considers a symplectic manifold $(P, \omega)$ with an action of a Lie group G on P. One assumes that the action of G on P admits a momentum map $J : P \rightarrow g^*$, $J(x) = \xi^*$ [$g^*$ is the dual of the Lie algebra $g$ of G], such that, for each vector $\xi \in g$, the quantity $J(\xi)(.) =< J, \xi > (.) : P \rightarrow R$ is the Hamiltonian function (constant of the motion) generating the symplectic action on P. The space ("level set" for the action of G on P) $J^{-1}(\xi) \subset P$ is a manifold except at points where the action of G has a nontrivial isotropy group (where at most quadratic singularities occur; see II for this problematic). If $G_\xi$ is the isotropy group of $\xi \in g$ with respect to the coadjoint action of G on $g^*$, then the space $J^{-1}(\xi)/G_\xi$ is a stratified symplectic manifold with at most quadratic singularities (the reduced phase space or the space of dynamical degrees of freedom).

For metric general relativity in the spatially compact case this programme has been carried out in Ref. [31]. If $M^4 = \Sigma \times R$ with $\Sigma$ a compact, orientable 3-manifold, if ${}^4g$ is a Lorentz metric on $M^4$ satisfying Einstein's equations and if $Riem\,({}^3g)$ is the set of Riemannian metrics on $\Sigma$, then the group G is $Diff\, M^4$ and the momentum map consists of the ADM supermomentum and superhamiltonian constraints $\phi = \{\tilde{\mathcal{H}}, {}^3\tilde{\mathcal{H}}^r\} : T^*Riem\,({}^3g) \rightarrow \Lambda^o_d \times \Lambda^1_d$ [the dual of the space of infinitesimal diffeomorphisms interpreted as the space of lapses and shifts since they are the coefficients of the constraints in the Dirac Hamiltonian], even if the action of $Diff\,\Sigma$ on the constraint set is not a true group action since there are structure functions (see Section V of I). The space of solutions of Einstein's equations is fibered over $\phi^{-1}(0) \in T^*Riem\,({}^3g)$, which is smooth at $({}^3g, {}^3\tilde{\Pi})$ if and only if the initial data $({}^3g, {}^3\tilde{\Pi})$ correspons to a solution ${}^3g$ with no Killing field. The reduced phase space $\phi^{-1}(0)/Diff\, M^4$ has been constructed [31] and turns out to be a stratified symplectic ILH manifold.

In the spatially asymptotically flat case, the group G becomes the group of those diffeomorphisms which preserve the conditions for asymptotic flatness and the nature of this group depends strongly on the precise asymptotic conditions. Apart from the compactification schemes of Geroch [15] and of Ashtekar-Hansen [16], 3 main types of asymptotic conditions have been studied: i) the "finite energy" condition of O'Murchadha [32]; ii) the

York "quasi isotropic" (QI) gauge conditions [33] (see Appendix B); iii) the conditions of the type introduced by Regge-Teitelboim [5] with the "parity conditions" further refined by Beig-O'Murchadha [6] (see the next Section) plus the gauge conditions of maximal slices and "3-harmonic" asymptotic coordinates [their existence was shown in Ref. [32]]. These 3 types of asymptotic conditions have quite different properties. In the case of the finite energy conditions of i), one finds that the group which leaves the asymptotic conditions invariant is a semidirect product $S \,|\!\times L$, where L is the Lorentz group and S consists of diffeomorphisms $\eta$ such that roughly $D^2\eta \in L^2$ [i.e. S contains space- and time- translations]. Under these conditions, it does not appear to be possible to talk about Hamiltonian dynamics. For a general element of the Lie algebra of $S \,|\!\times L$, the corresponding momentum integral does not converge, although for the special case of space- and time-translations the ADM 4-momentum is well defined.

York QI gauge conditions of Ref. [33] have the desirable feature that "no supertranslations" are allowed, but a more detailed analysis reveals that without extra conditions, the transformations corresponding to boosts are not well behaved; in any case, the QI asymptotic conditions do not give a well defined angular and boost momentum and therefore are suitable only for the study of diffeomorphisms asymptotic to space- and time- translations.

To get a well defined momentum for rotations and boosts, Anderson defines asymptotic conditions which contain the parity conditions of Ref. [6], but he replaces the 3-harmonic coordinates used in this paper with York's QI conditions. The space of transformations $Diff_P\, M^4$ which leaves invariant the space of solutions of the Einstein equations satisfying the parity conditions is a semidirect product $Diff_P\, M^4 = Diff_S\, M^4 \,|\!\times P$, where P is the Poincaré group and $Diff_S\, M^4$ denotes the space of diffeomorphisms which are asymptotic to supertranslations, which in this case are O(1) with odd leading term. When the QI conditions are added, the $Diff_S\, M^4$ part is restricted to $Diff_I\, M^4$, the space of diffeomorphisms which tend to the identity at spatial infinity [this result cannot be obtained with the finite energy conditions [32] or from boost theorems [34]]. In this way one obtains a realization of Bergmann's criticism [3] of general covariance: the group of coordinate transformations is restricted to contain an invariant Poincaré subgroup plus asymptotically trivial diffeomorphisms (similar to the gauge transformations of electromagnetism). It can be shown that, when using the parity conditions, the lapses and shifts corresponding to the group of supertranslations S have "zero momentum". Thus, assuming the QI conditions, the ADM momentum appears as the momentum map with respect to the Poincaré group. Note that the classical form of the ADM momentum is correct only using the restrictive assumption of parity conditions, which are nontrivial restrictions "not only" on the gauge freedom "but also" on the asymptotic dynamical degrees of freedom of the gravitational field [this happens also with Ashtekar-Hansen asymptotic condition on the Weyl tensor]. Call the total momentum map $\phi_E = \phi + E : T^*Riem\,({}^3g) \rightarrow (\Lambda^o_d \times \Lambda^1_d) \times p^*$, where $\Lambda^o_d \times \Lambda^1_d$ is the 3+1 version of the dual of the Lie algebra of $Diff_I\, M^4$ and $p^*$ is the dual of the Poincaré Lie algebra p; E will consist of certain integrals over spheres at spatial infinity. One now expects (assuming the QI conditions) that: i) $\phi^{-1}(0)/Diff_I\, M^4$ is a symplectic manifold (no isometry can be in $Diff_I\, M^4$); ii) for $\xi \in p^*$, the spaces $\phi^{-1}_E(0 \times \xi)$ and $\phi^{-1}_E(0 \times \xi)/Diff_I\, M^4$ are manifolds except at points corresponding to flat space or spaces with rotational symmetries [$Diff_{P_\xi}\, M^4 = Diff_I\, M^4 \times P_\xi$, where $P_\xi$ denotes the isotropy group of $\xi$ in P].

By assuming the validity of the conjecture on global existence of solutions of Einstein's

equations and of maximal slicing (in the framework of York reduction; see Appendix C of II) and working with Sobolev spaces with "radial smoothness", Anderson demonstrates a "slice" theorem, according to which, assumed the parity and QI conditions [which exclude the logarithmic translations of Bergmann [20]], for every solution ${}^3g_o$ of Einstein's equations one has that: i) the gauge orbit of ${}^3g_o$ is a closed $C^1$ embedded submanifold of the manifold of solutions; ii) there exists a submanifold containing ${}^3g_o$ which is a "slice for the action of $Diff_I\, M^4$" (see Appendix B of Anderson's paper for the definition of slice).

York's QI conditions should be viewed as a slice condition which fixes part of the gauge freedom at spatial infinity: i) the $O(1/r^2)$ part of the trace of ${}^3\tilde{\Pi}^{rs}$ must vanish; ii) if ${}^3g = {}^3f + {}^3h$ (${}^3f$ is a flat metric) and if ${}^3h = {}^3h_{TT} = {}^3h_T + L_f(W)$ is the York decomposition of ${}^3h$ with respect to ${}^3f$ (see Appendix C of II), then the O(1) part of the longitudinal quantity W must vanish. In this way, one selects a QI asymptotically flat metric ${}^3g_{QI}$ and a "preferred frame at spatial infinity" (remember Bergmann's criticism to general covariance [3], quoted in the Conclusions of II), i.e. preferred spacelike hypersurfaces $\Sigma_{QI}$ [they can be mapped onto the space $\Sigma_{\mathcal{K}}$ of cross sections of the unit timelike hyperboloid $\mathcal{K}$ (with which Beig [21] completes $M^4$ at spatial infinity), corresponding to the set of intersections of $\mathcal{K}$ by spatial hyperplanes in $R^4$], as shown in a lemma of Ref. [30].

Then, in the Anderson paper [30] there are comments on the open problem of the validity of the conjecture of existence of maximal slicings [see Appendix C of II; in this case the QI conditions are natural], due to the discovery of topological obstructions which invalidate the demonstrations of certain theorems. Anderson [30] showed that to put real control on boosts and rotations, one should add the parity conditions to the weighted Sobolev spaces [this reduces the class of solutions of Einstein's solutions]

Let us add other results connected with the previous problematic and with the existence of the ADM Lorentz generators.

In Ref. [34] on the boost problem in general relativity, Christodoulou and Murchadha show (using weighted Sobolev spaces) that a very large class of asymptotically flat initial data for Einstein's equations have a development which includes complete spacelike surfaces boosted relative to the initial surface. Furthermore, the asymptotic fall off [${}^3g - {}^3f \in W^{2,s,\delta+1/2}(\Sigma)$, ${}^3K \in W^{2,s-1,\delta+1/2}(\Sigma)$, $s \geq 4$, $\delta > -2$] is preserved along these boosted surfaces and there exist a global system of harmonic coordinates on such a development.

As noted in Ref. [18], the results of Ref. [34] suffice to establish the existence of a large class off spacetimes which are asymptotically flat at $i^o$ (in the sense of Ref. [16]) in all spacelike directions along a family of Cauchy surfaces related to one another by "finite" boosts (it is hoped that new results will allow to put control also on "infinite" boosts). The situation is unsettled with regard the existence of spacetimes admitting both $i^o$ (in the sense of Ref. [16]) as well as smooth $\mathcal{I}^{\pm}$.

In Ref. [32], by making use of the previous paper [34] and of weighted Sobolev spaces, it is shown that the 1/r behaviour of the metric in the asymptotically flat case (used in the boost and positive energy theorems) can be relaxed to $r^{-1/2+\epsilon}$ without destroying the existence of a well defined, conserved, Lorentz-covariant,timelike, future-pointing energy-momentum 4-vector. The 1/r behaviour is connected to the fact that the ADM conserved quantities are expressed as surface integrals at infinity which must be finite. However, one can use the Gauss theorem to turn the surface integrals into volume integrals: the leading term in the volume integral expression vanishes, due to the constraints, and thus the volume

integral expressions for the energy- momentum converges (on the constraint manifold) using the weaker fall off condition $r^{-1/2+\epsilon}$ [in electrodynamics one has $Q = e\int_{\partial V} \vec{E}\cdot d\vec{S} \stackrel{\circ}{=} Q' = e\int_V div\, \vec{E} d^3x \approx \int_V \rho d^3x$ ($Q$ and $Q'$ are the strong and weak improper charges, see the paper c) in Ref. [35]); so that $\vec{E} = O(1/r^2)$ required by the finiteness of Q is replaced by the $\vec{E}$-independent requirement $\rho = O(1/r^3)$ by using the Gauss law constraint]. The falloff conditions of this paper are ${}^3g - {}^3f \in W^{2,s,\delta}(R^3)$, ${}^3K \in W^{2,s-1,\delta+1}(R^3)$, $s \geq 4$, $\delta > -1$.

In Ref. [36], Chruściel says that for asymptotically flat metrics ${}^3g = {}^3f + O(r^{-\alpha})$, $\frac{1}{2} < \alpha \leq 1$, it is not proved the "asymptotic symmetry conjecture" that, given any two coordinate systems of the previous type, all twice-differentiable coordinate transformations preserving these boundary conditions are of type $y^\mu = \Lambda^\mu{}_\nu x^\nu + \zeta^\mu$ [a Lorentz transformation + a supertranslation $\zeta = O(r^{1-\alpha})$]: this would be needed for the ADM 4-momentum to be Lorentz covariant. By defining $P_\mu$ in terms of Cauchy data on a 3-end N [a spacelike 3-surface $\Sigma$ minus a ball] on which ${}^3g = {}^3f + O(r^{-\alpha})$, one can evaluate the invariant mass $m(N) = \sqrt{\epsilon P^\mu P_\mu}$. Then, provided the hypersurfaces $x^o = const.$ $(N_1)$, $y^o = const.$ $(N_2)$, lie within a "finite" boost of each other or if the metric is a no-radiation metric, one can show the validity of the "invariant mass conjecture" $m(N_1) = m(N_2)$ for metrics satisfying vacuum Einstein equations. The main limitation is the lack of knowledge of long-time behaviour of Einstein's equations. Ashtekar-Hansen and Beig-O'Murchadha requirements are much stronger and restrictive than what is compatible with Einstein's equations.

Since there is no agreement among the various viewpoints on the coordinate-independent definition of asymptotic flatness at spatial infinity, since we are interested in the coupling of general relativity to the standard SU(3)xSU(2)XU(1) model and since we wish to recover the theory in Minkowski spacetime if we put G=0 (the deparametrization problem of general relativity, only partially solved in Ref. [37] by using coordinate gauge conditions and studied in Ref. [38]), in this paper we shall use a coordinate-dependent approach and we shall work in the framework of Refs. [5,6], in which supertranslations may be eliminated and there is a well defined Poincaré asymptotic symmetry group due to the choice of the boundary conditions and of a certain class of gauge-fixings for the supermomentum constraints of metric gravity. This will also be connected with Bergmann's remarks [3] on the existence of preferred coordinate systems breaking general covariance (see the Conclusions of II and next Sections).

In particular, the chosen boundary conditions and gauge-fixings will imply an angle (i.e. direction)-independent asymptotic limit of the canonical variables, just as it is needed in Yang-Mills theory to have well defined covariant non-Abelian charges [39,40] [as shown in Ref. [40], one needs a set of Hamiltonian (not manifestly covariant except in the reformulation on spacelike hypersurfaces) boundary conditions both for the fields and the gauge transformations in the Hamiltonian gauge group $\bar{\mathcal{G}}$, implying angle-independent limits at spatial infinity; it is also suggested that the elimination of Gribov ambiguity requires the use of the following weighted Sobolev spaces [41] : $\vec{A}_a, \vec{E}_a \in W^{p,s-1,\delta-1}$, $\vec{B}_a \in W^{p,s-2,\delta+2}$, $\bar{\mathcal{G}} \in W^{p,s,\delta}$, with $p > 3$, $s \geq 3$, $0 \leq \delta \leq 1 - \frac{3}{p}$]. This is an important point for a future unified description of general relativity and of the standard model.

In particular, following Ref. [42] (see also Appendix A), we will assume that at spatial infinity there is a 3-surface $S_\infty$ [not necessarily a timelike hyperboloid at this stage of development], which intersects orthogonally the Cauchy surfaces $\Sigma_\tau$ [their normals $l^\mu(\tau,\vec{\sigma})$ at spatial infinity, $l^\mu_{(\infty)\Sigma}$, are tangent to $S_\infty$] along 2-surfaces $S^2_{\tau,\infty}$. Since we will identify special

families of hypersurfaces $\Sigma_\tau$ asymptotic to Minkowski hyperplanes at spatial infinity, these families can be mapped onto the space of cross sections of the unit timelike hyperboloid by using the quoted Anderson's lemma [30].

## III. POINCARÉ CHARGES IN METRIC GRAVITY.

Before discussing the asymptotic Poincaré charges, let us summarize what is known about the non Abelian charges and the Hamiltonian group of gauge transformations for Yang-Mills theory.

As emphasized in Ref. [40], in the Hamiltonian formulation of a gauge theory one has to make a choice of the boundary conditions of the canonical variables and of the parameters of the gauge transformations [the infinitesimal ones are generated by the first class constraints of the theory] in such a way to give a meaning to integrations by parts, to the functional derivatives (and therefore to Poisson brackets) and to the "proper" gauge transformations connected with the identity [the "improper" ones, including the "rigid or global or first kind" gauge transformations related to the non-Abelian charges, have to be treated separately; when there are topological numbers like winding number, they label disjoint sectors of gauge transformations and one speaks of "large" gauge transformations]. In particular, the boundary conditions must be such that the variation of the final Dirac Hamiltonian $H_D$ must be linear in the variations of the canonical variables [the coefficients are the Dirac-Hamilton equations of motion] and this may require a redefinition of $H_D$, namely $H_D$ has to be replaced by $\tilde{H}_D = H_D + H_\infty$, where $H_\infty$ is a suitable integral on the surface at spatial infinity. When this is accomplished, one has a good definition of functional derivatives and Poisson brackets. Then, one must consider the most general generator of gauge transformations of the theory (it includes $H_D$ as a special case), in which there are arbitrary functions (parametrizing infinitesimal gauge transformations) in front of all the first class constraints and not only in front of the primary ones. Also the variations of this generator must be linear in the variations of the canonical variables: this implies that all the surface terms coming from integration by parts must vanish with the given boundary conditions on the canonical variables or must be compensated by the variation of $H_\infty$. In this way, one gets boundary conditions on the parameters of the infinitesimal gauge transformations identifying the "proper" ones, which transform the canonical variables among themselves without altering their boundary conditions [the symplectic vector fields associated with the proper gauge transformations map the function space of the canonical variables into itself]. Let us remark that in this way one is defining Hamiltonian boundary conditions which are not manifestly covariant; however, in Minkowski spacetime a Wigner covariant formulation is obtained by reformulating the theory on spacelike hypersurfaces [43,44] and then restricting it to spacelike hyperplanes.

In the Yang-Mills case [40], with the Hamiltonian gauge transformations restricted to go to the identity in an angle-independent way at spatial infinity , so to have well defined covariant non-Abelian charges, the "proper" gauge transformations are those which are connected to the identity and generated by the Gauss law first class constraints at the infinitesimal level. The "improper" ones are a priori of four types:

i) "global or rigid or first kind" ones (the gauge parameters fields tend to constant at spatial infinity) connected with the group G (isomorphic to the structure group of the Yang-Mills principal bundle) generated by the "non-Abelian charges";
ii) the global or rigid ones in the "center of the gauge group G" [triality when G=SU(3)];
iii) gauge transformations with non-vanishing winding number $n \in Z$ ("large" gauge trans-

formations not connected with the identity; zeroth homotopy group of the gauge group); iv) other "improper non rigid" gauge transformations. Since this last type of gauge transformations does not play any role in Yang-Mills dynamics, it was assumed [40] that the choice of the function space for the gauge parameter fields $\alpha_a(\tau,\vec{\sigma})$ (describing the component of the gauge group connected with the identity) be such that for $r\,\rightarrow\infty$ one has

$$\alpha_a(\tau,\vec{\sigma})\,\rightarrow\alpha_a^{(rigid)}+\alpha_a^{(proper)}(\tau,\vec{\sigma})$$

with constant $\alpha_a^{(rigid)}$ and with $\alpha_a^{(proper)}(\tau,\vec{\sigma})$ tending to zero in a direction-independent way.

In metric gravity, the Hamiltonian gauge group is not connected with a principal bundle (it contains 3-diffeomorphisms and its algebra has structure functions and not structure constants). However, the asymptotic Poincaré charges (not uniquely defined when supertranslations are allowed) and the eventuality of supertranslations (whose generators, the "supertranslation charges", should vanish, i.e. have "zero momentum" according to Anderson [30], when the boundary conditions have well defined parity properties) are the counterpart of the Yang-Mills non-Abelian charges and also of the Abelian electric charge. While the electric charge is a physical observable, the hypothesis of quark confinement requires the existence only of color singlets, namely i) physical observables must commute with the non-Abelian charges; ii) the SU(3) color charges of isolated systems have to vanish themselves. In both cases, inside local quantum field theory, one speaks of superselection sectors determined by the charges and does not consider them as generators of gauge transformations.

In Ref. [45] the same possibility is opened for the asymptotic Poincaré charges of asymptotically flat metric gravity:

i) in the usual interpretation [46] some observer is assumed to sit at or just outside the boundary at spatial infinity but he is not explicitly included in the action functional; this observer merely supplies a coordinate chart on the boundaries (perhaps, through his 'parametrization clock'), which we may use to fix the gauge of our system at the boundary (the asymptotic lapse function; see later on); if one wishes, this external observer may construct his clock to yield zero Poincaré charges (so to recover a Machian interpretation [47] also in noncompact universes with boundary; there is a strong similarity with the results of Einstein-Wheeler cosmology [48], based on a closed compact universe without boundaries, for which Poincaré charges are not defined), in which case every connection with particle physics is lost;
ii) Marolf's proposal [45] is to consider the system in isolation without the utilization of any structure outside the boundary at spatial infinity and to consider, at the quantum level, superselection rules for the asymptotic Poincaré Casimirs, in particular for the ADM invariant mass [see Refs. [49] for similar conclusions from different motivations].

In Ref. [50], also Giulini considers a matter of physical interpretation whether all 3-diffeomorphisms of $\Sigma_\tau$ into itself must be considered as gauge transformations. In the asymptotically flat open case, there is in this paper a discussion of "large" diffeomorphisms, but the gauge transformations generated by the superhamiltonian constraint are not considered; after a 1-point compactification $\bar{\Sigma}_\tau$ of $\Sigma_\tau$, there is a study of the quotient space $Riem\,\bar{\Sigma}_\tau/Diff_F\,\bar{\Sigma}_\tau$, where $Diff_F\,\bar{\Sigma}_\tau$ are those 3-diffeomorphisms whose pullback goes to

the identity at spatial infinity (the point of compactification) where a "privileged oriented frame" is chosen. The Poincaré charges are not considered as generators of gauge transformations; instead, there is a study of the decomposition of $\bar{\Sigma}_\tau$ into its prime factors as a 3-manifold, of the induced decomposition of $Diff_F\, \bar{\Sigma}_\tau$ and of the evaluation of the homotopy groups of $Diff_f\, \bar{\Sigma}_\tau$.

We shall take the point of view that the asymptotic Poincaré charges are not generators of gauge transformations like in Yang-Mills theory (the ADM energy will be the physical Hamiltonian for the evolution in $\tau$), that there are superselection sectors labelled by the asymptotic Poincaré Casimirs and that the parameters of the gauge transformations of ADM metric gravity have a clean separation between a rigid part (differently from Yang-Mills theory it has both a constant and a term linear in $\vec{\sigma}$) and a proper one restricted not to contain supertranslations [namely we assume the absence of "improper non-rigid" gauge transformations like in Yang-Mills theory].

Let us now define the "proper" gauge transformations of the ADM metric gravity, whose canonical formalism was reviewed in Section V of I [in Appendix A there are other properties of metric gravity deriving by the application of the second Noether theorem [35] to the ADM action]. In Refs. [51,52,42], as shown in Appendix A, it is noted that, in asymptotically flat spacetimes, the surface integrals arising in the transition from the Hilbert action to the ADM action and, then, from this to the ADM phase space action are connected with the ADM energy-momentum of the gravitational field of the linearized theory of metric gravity [4], if the lapse and shift functions have certain asymptotic behaviours at spatial infinity. Extra complications for the differentiability of the ADM canonical Hamiltonian come from the presence of the second spatial derivatives of the 3-metric inside the ${}^3R$ term of the superhamiltonian constraint .

Regge and Teitelboim [5] gave a set of boundary conditions for the ADM canonical variables ${}^3g_{rs}(\tau,\vec{\sigma})$, ${}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma})$, so that it is possible to define 10 surface integrals associated with the conserved Poincaré charges of the spacetime (the translation charges are the ADM energy-momentum) and to show that the functional derivatives and Poisson brackets are well defined in metric gravity; however, there is no statement about gauge transformations and supertranslations in this paper.

A more complete analysis, including also a discussion of supertranslations in the ADM canonical formalism, has been given by Beig and O'Murchadha [6] (extended to Ashtekar's formalism in Ref. [53]). They consider 3-manifolds $\Sigma_\tau$ diffeomorphic to $R^3$ as in our case, so that there exist "global coordinate systems". If $\{\sigma^{\check{r}}\}$ is one of these global coordinate systems on $\Sigma_\tau$, the 3-metric ${}^3g_{\check{r}\check{s}}(\tau,\sigma^{\check{t}})$ [evaluated in this coordinate system] is assumed asymptotically Euclidean in the following sense: if $r=\sqrt{\delta_{\check{r}\check{s}}\sigma^{\check{r}}\sigma^{\check{s}}}$ [one could put $r=\sqrt{{}^3g_{\check{r}\check{s}}\sigma^{\check{r}}\sigma^{\check{s}}}$ and get the same kind of decomposition], then one assumes

$$
\begin{aligned}
{}^3g_{\check{r}\check{s}}(\tau,\vec{\sigma}) &= \delta_{\check{r}\check{s}}+\frac{1}{r}\,{}^3s_{\check{r}\check{s}}(\tau,\frac{\sigma^{\check{n}}}{r})+{}^3h_{\check{r}\check{s}}(\tau,\vec{\sigma}),\quad r\,\rightarrow\infty,\\
&\quad {}^3s_{\check{r}\check{s}}(\tau,\frac{\sigma^{\check{n}}}{r})={}^3s_{\check{r}\check{s}}(\tau,-\frac{\sigma^{\check{n}}}{r}),\quad EVEN\,PARITY,\\
&\quad {}^3h_{\check{r}\check{s}}(\tau,\vec{\sigma})=O((r^{-(1+\epsilon)}),\quad \epsilon>0,\,for\,r\,\rightarrow\infty,\\
&\quad \partial_{\check{u}}\,{}^3h_{\check{r}\check{s}}(\tau,\vec{\sigma})=O(r^{-(2+\epsilon)}).
\end{aligned}
\tag{2}
$$

The functions ${}^3s_{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r})$ are $C^\infty$ on the sphere $S^2_{\tau,\infty}$ at spatial infinity; if they would be of odd parity, the ADM energy would vanish. The difference ${}^3g_{\check{r}\check{s}}(\tau, \vec{\sigma}) - \delta_{\check{r}\check{s}}$ cannot fall off faster that 1/r, because otherwise the ADM energy would be zero and the positivity energy theorem [54] would imply that the only solution of the constraints is flat spacetime. For the ADM momentum one assumes the following boundary conditions

$$
{}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau, \vec{\sigma}) = \frac{1}{r^2}\, {}^3t^{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) + {}^3k^{\check{r}\check{s}}(\tau, \vec{\sigma}), \quad r \rightarrow \infty,
$$

$$
\begin{aligned}
&{}^3t^{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) = -{}^3t^{\check{r}\check{s}}(\tau, -\frac{\sigma^{\check{n}}}{r}), \quad ODD\, PARITY,\\
&{}^3k^{\check{r}\check{s}}(\tau, \vec{\sigma}) = O(r^{-(2+\epsilon)}), \quad \epsilon > 0, \quad r \rightarrow \infty.
\end{aligned}
\tag{3}
$$

If ${}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau, \vec{\sigma})$ were to fall off faster than $1/r^2$, the ADM linear momentum would vanish and we could not consider Lorentz transformations. In this way, the integral $\int_{\Sigma_\tau} d^3\sigma [{}^3\tilde{\Pi}^{\check{r}\check{s}} \delta\, {}^3g_{\check{r}\check{s}}](\tau, \vec{\sigma})$ is well defined and finite [since the integrand is of order $O(r^{-3})$, a possible logarithmic divergence is avoided due to the odd parity of ${}^3t^{\check{r}\check{s}}$].

These boundary conditions imply that functional derivatives and Poisson brackets are well defined [6] [in a more rigorous treatment one should use appropriate weighted Sobolev spaces].

The supermomentum and superhamiltonian constraints [see Eqs.(79) of I and Appendix A] ${}^3\tilde{\mathcal{H}}^{\check{r}}(\tau, \vec{\sigma}) \approx 0$ and $\tilde{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$, are even functions of $\vec{\sigma}$ of order $O(r^{-3})$. Their smeared version with the lapse and shift functions, appearing in the canonical Hamiltonian $H_{(c)ADM} = \int d^3\sigma [N\tilde{\mathcal{H}} + N_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau, \vec{\sigma})$, will give a finite and differentiable $H_{(c)ADM}$ if we set [6]

$$
\begin{aligned}
N(\tau, \vec{\sigma}) &= m(\tau, \vec{\sigma}) = s(\tau, \vec{\sigma}) + n(\tau, \vec{\sigma}) = k(\tau, \frac{\sigma^{\check{n}}}{r}) + O(r^{-\epsilon}), \quad \epsilon > 0, \quad r \rightarrow \infty,\\
N_{\check{r}}(\tau, \vec{\sigma}) &= m_{\check{r}}(\tau, \vec{\sigma}) = s_{\check{r}}(\tau, \vec{\sigma}) + n_{\check{r}}(\tau, \vec{\sigma}) = k_{\check{r}}(\tau, \frac{\sigma^{\check{n}}}{r}) + O(r^{-\epsilon}),
\end{aligned}
$$

$$
\begin{aligned}
s(\tau, \vec{\sigma}) &= k(\tau, \frac{\sigma^{\check{n}}}{r}) = -k(\tau, \frac{\sigma^{\check{n}}}{r}), \quad ODD\, PARITY,\\
s_{\check{r}}(\tau, \vec{\sigma}) &= k_{\check{r}}(\tau, \frac{\sigma^{\check{n}}}{r}) = -k_{\check{r}}(\tau, \frac{\sigma^{\check{n}}}{r}), \qquad ODD\, PARITY.
\end{aligned}
\tag{4}
$$

With these boundary conditions one gets differentiability, i.e. $\delta H_{(c)ADM}$ is linear in $\delta\, {}^3g_{\check{r}\check{s}}$ and $\delta\, {}^3\tilde{\Pi}^{\check{r}\check{s}}$, with the coefficients being the Dirac-Hamilton equations of metric gravity. Therefore, since N and $N_{\check{r}}$ are a special case of the parameter fields of the most general infinitesimal gauge transformations generated by the first class constraints $\tilde{\mathcal{H}}$, ${}^3\tilde{\mathcal{H}}^{\check{r}}$, with generator $G = \int d^3\sigma [\alpha\tilde{\mathcal{H}} + \alpha_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau, \vec{\sigma})$, the "proper" gauge transformations preserving Eqs.(2) and (3) have the multiplier fields $\alpha(\tau, \vec{\sigma})$ and $\alpha_{\check{r}}(\tau, \vec{\sigma})$ with the same boundary conditions (4) of $m(\tau, \vec{\sigma})$ and $m_{\check{r}}(\tau, \vec{\sigma})$. Then, the Hamilton equations imply that also the Dirac multipliers $\lambda_N(\tau, \vec{\sigma})$ and $\lambda^{\vec{N}}_{\check{r}}(\tau, \vec{\sigma})$ have these boundary conditions [$\lambda_N \circeq \delta N$, $\lambda^{\vec{N}}_{\check{r}} \circeq \delta N_{\check{r}}$]. Instead, the momenta $\tilde{\Pi}^N(\tau, \vec{\sigma})$ and $\tilde{\Pi}^{\check{r}}_{\vec{N}}(\tau, \vec{\sigma})$, conjugate to N and $N_{\check{r}}$, must be of $O(r^{-(3+\epsilon)})$ to have $H_{(D)ADM}$ finite.

The angle-dependent functions $s(\tau, \vec{\sigma}) = k(\tau, \frac{\sigma^{\check{n}}}{r})$ and $s_{\check{r}}(\tau, \vec{\sigma}) = k_{\check{r}}(\tau, \frac{\sigma^{\check{n}}}{r})$ on $S^2_{\tau,\infty}$ (boundary of $\Sigma_\tau$ at spatial infinity) are "odd time and space supertranslations". The piece

$\int d^3\sigma [s\, \tilde{\mathcal{H}} + s_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau, \vec{\sigma}) \approx 0$ of the Dirac Hamiltonian is the Hamiltonian generator of supertranslations (the "zero momentum" of supertranslations of Ref. [30]). Their contribution to gauge transformations is to alter the angle-dependent asymptotic terms ${}^3s_{\check{r}\check{s}}$ and ${}^3t^{\check{r}\check{s}}$ in ${}^3g_{\check{r}\check{s}}$ and ${}^3\tilde{\Pi}^{\check{r}\check{s}}$.

With $N = m$, $N_{\check{r}} = m_{\check{r}}$ one can verify the validity of the smeared form of Eqs.(81) of I:

$$\begin{aligned} &\{H_{(c)ADM}[m_1, m_1^{\check{r}}], H_{(c)ADM}[m_2, m_2^{\check{r}}]\} = \\ &= H_{(c)ADM}[m_2^{\check{r}}\, {}^3\nabla_{\check{r}} m_1 - m_1^{\check{r}}\, {}^3\nabla_{\check{r}} m_2,\, \mathcal{L}_{\vec{m}_2}\, m_1^{\check{r}} + m_2\, {}^3\nabla^{\check{r}} m_1 - m_1\, {}^3\nabla^{\check{r}} m_2], \end{aligned} \tag{5}$$

with $m^{\check{r}} = {}^3g^{\check{r}\check{s}} m_{\check{s}}$ and with $H_{(c)ADM}[m, m^{\check{r}}] = \int d^3\sigma [m\tilde{\mathcal{H}} + m^{\check{r}}\, {}^3\tilde{\mathcal{H}}_{\check{r}}](\tau, \vec{\sigma}) = \int d^3\sigma [m\tilde{\mathcal{H}} + m_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau, \vec{\sigma})$.

When the functions $N(\tau, \vec{\sigma})$ and $N_{\check{r}}(\tau, \vec{\sigma})$ [and also $\alpha(\tau, \vec{\sigma})$, $\alpha_{\check{r}}(\tau, \vec{\sigma})$] do not have the asymptotic behaviour of $m(\tau, \vec{\sigma})$ and $m_{\check{r}}(\tau, \vec{\sigma})$ respectively, one speaks of "improper" gauge transformations, because $H_{(D)ADM}$ is not differentiable even at the constraint hypersurface [in Ref. [6] there is a non-rigorous proof that this hypersurface is a manifold; this can be possible only if the spacetime has no isometries].

At this point one has identified:

a) Certain coordinate systems on the spacelike 3-surface $\Sigma_\tau$, which hopefully define a minimal atlas $\mathcal{C}_\tau$ for the spacelike hypersurfaces $\Sigma_\tau$ foliating the asymptotically flat spacetime $M^4$. With the $\mathcal{C}_\tau$'s and the parameter $\tau$ as $\Sigma_\tau$-adapted coordinates of $M^4$ one should build an atlas $\mathcal{C}$ of allowed coordinate systems of $M^4$.

b) A set of boundary conditions on the fields on $\Sigma_\tau$ (i.e. a function space for them) ensuring that the 3-metric on $\Sigma_\tau$ is asymptotically Euclidean in this minimal atlas (modulo 3-diffeomorphisms, see the next point).

c) A set of "proper" gauge transformations generated infinitesimally by the first class constraints, which leave the fields on $\Sigma_\tau$ in the chosen function space. Since the gauge transformations generated by the supermomentum constraints ${}^3\tilde{\mathcal{H}}^{\check{r}}(\tau, \vec{\sigma}) \approx 0$ are the lift to the space of the tensor fields on $\Sigma_\tau$ (which contains the phase space of metric gravity) of the 3-diffeomorphisms $Diff\, \Sigma_\tau$ of $\Sigma_\tau$ into itself, the restriction of $N(\tau, \vec{\sigma})$, $N_{\check{r}}(\tau, \vec{\sigma})$ to $m(\tau, \vec{\sigma})$, $m_{\check{r}}(\tau, \vec{\sigma})$, ensures that these 3-diffeomorphisms are restricted to be compatible with the chosen minimal atlas for $\Sigma_\tau$ [this is the problem of the coordinate transformations preserving Eq.(2)].

More difficult is the interpretation [55,56] of the gauge transformations generated by the superhamiltonian constraint $\tilde{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$, which are the phase space adaptation of the time diffeomorphisms of $M^4$ (leaving invariant the Hilbert action $S_H$) after the removal of the surface term connecting the Hilbert action $S_H$ to the ADM action $S_{ADM}$ [which is quasi-invariant under the gauge transformations generated by $\tilde{\mathcal{H}}(\tau, \vec{\sigma})$ but not under $Diff\, M^4$, as shown in Appendix A]. Since the superhamiltonian constraint determines the conformal factor of the 3-metric (see the Conclusions of II and Section VI), the gauge transformations generated by $\tilde{\mathcal{H}}(\tau, \vec{\sigma})$ will induce a change of the momentum conjugate to this conformal factor, namely of the extrinsic curvature of $\Sigma_\tau$, so that they generate the transitions from one allowed 3+1 splitting of $M^4$ to another allowed one. This suggests that, in absence of supertranslations, the functions $N$, $\alpha$, $\lambda_N$, should go like $O(r^{-(2+\epsilon)})$ and not like $O(r^{-\epsilon})$ (in the case of proper gauge transformations).

Let us remark at this point that the addition of gauge-fixing constraints to the superhamiltonian and supermomentum constraints (see the end of Section V of I) must happen

in the chosen function space for the fields on $\Sigma_\tau$. Therefore, the time constancy of these gauge-fixings will generate secondary gauge-fixing constraints for the restricted lapse and shift functions $m(\tau,\vec{\sigma})$, $m_{\check{r}}(\tau,\vec{\sigma})$.

By using the original ADM results [4], Regge and Teitelboim [5] wrote the expression of the ten conserved Poincaré charges, by allowing the functions $N(\tau,\vec{\sigma})$, $N_{\check{r}}(\tau,\vec{\sigma})$, to have a linear behaviour in $\vec{\sigma}$ for $r \rightarrow \infty$. These charges are surface integrals at spatial infinity, which have to added to the Dirac Hamiltonian so that it becomes differentiable. Therefore, also the parameter fields $\alpha(\tau,\vec{\sigma})$, $\alpha_{\check{r}}(\tau,\vec{\sigma})$, of arbitrary (also improper) gauge transformations should acquire this behaviour. In this way one is enlarging the allowed 3-diffeomorphisms of $\Sigma_\tau$ into itself (this would require an enlargement of the minimal atlas of $\Sigma_\tau$) and one should change the function space of the tensor fields ${}^3g_{\check{r}\check{s}}$, ${}^3\tilde{\Pi}^{\check{r}\check{s}}$. The added "Poincaré transformations at infinity" generated by the ten charges are often considered as "extra improper gauge transformations". However, if one takes Marolf's viewpoint that the Poincaré charges are not generators of improper gauge transformations but that they determine superselection sectors, one has not to enlarge the allowed 3-diffeomorphisms and the minimal atlas of $\Sigma_\tau$.

These results, the wish to have a clear separation between proper and improper gauge transformations (like in Yang-Mills theory) and the hope to solve the deparametrization problem of general relativity suggest that it is reasonable to break the lapse and shift functions in the parts associated with the proper and improper gauge transformations respectively (with no "improper non rigid" gauge transformations like in Yang-Mills theory)

$$\begin{aligned} N(\tau,\vec{\sigma}) &= N_{(as)}(\tau,\vec{\sigma}) + m(\tau,\vec{\sigma}),\\ N_{\check{r}}(\tau,\vec{\sigma}) &= N_{(as)\check{r}}(\tau,\vec{\sigma}) + m_{\check{r}}(\tau,\vec{\sigma}), \end{aligned} \tag{6}$$

and then to assume that the improper parts $N_{(as)}$, $N_{(as)r}$, behave as the lapse and shift functions associated with spacelike hyperplanes in Minkowski spacetime, as already anticipated in II.

As shown in II, there are two independent definitions of lapse and shift functions which can be associated with a 3+1 splitting of Minkowski spacetime by means of a foliation with spacelike hypersurfaces. However, for foliations with spacelike hyperplanes the two definitions coincide and produce the following result [$(\mu)$ are flat Minkowski indices in rectangular coordinates; $\tilde{\lambda}_{(\mu)}(\tau)$, $\tilde{\lambda}_{(\mu)(\nu)}(\tau)$, are arbitrary Dirac multipliers]

$$\begin{aligned} N_{(flat)}(\tau,\vec{\sigma}) &= N_{[z](flat)}(\tau,\vec{\sigma}) = -\tilde{\lambda}_{(\mu)}(\tau) l^{(\mu)} - l^{(\mu)}\tilde{\lambda}_{(\mu)(\nu)}(\tau) b^{(\nu)}_{\check{s}}(\tau)\sigma^{\check{s}},\\ N_{(flat)}(\tau,\vec{\sigma}) &= N_{[z](flat)\check{r}}(\tau,\vec{\sigma}) = -\tilde{\lambda}_{(\mu)}(\tau) b^{(\mu)}_{\check{r}}(\tau) - b^{(\mu)}_{\check{r}}(\tau)\tilde{\lambda}_{(\mu)(\nu)}(\tau) b^{(\nu)}_{\check{s}}(\tau)\sigma^{\check{s}}. \end{aligned} \tag{7}$$

In Ref. [8] and in the book in Ref. [9] (see also Ref. [5]), Dirac introduced asymptotic Minkowski rectangular coordinates

$$z^{(\mu)}_{(\infty)}(\tau,\vec{\sigma}) = x^{(\mu)}_{(\infty)}(\tau) + b^{(\mu)}_{(\infty)\,\check{r}}(\tau)\sigma^{\check{r}} \tag{8}$$

in $M^4$ at spatial infinity $S_\infty = \cup_\tau S^2_{\tau,\infty}$ [here $\{\sigma^{\check{r}}\}$ are the previous global coordinate systems of the atlas $\mathcal{C}_\tau$ of $\Sigma_\tau$, not matching the spatial coordinates $z^{(i)}_{(\infty)}(\tau,\vec{\sigma})$]. For each value of $\tau$, the coordinates $x^{(\mu)}_{(\infty)}(\tau)$ labels a point, near spatial infinity chosen as origin. On it there is a flat tetrad $b^{(\mu)}_{(\infty)\,A}(\tau) = (\, l^{(\mu)}_{(\infty)} = b^{(\mu)}_{(\infty)\,\tau} = \epsilon^{(\mu)}{}_{(\alpha)(\beta)(\gamma)} b^{(\alpha)}_{(\infty)\,\check{1}}(\tau) b^{(\beta)}_{(\infty)\,\check{2}}(\tau) b^{(\gamma)}_{(\infty)\,\check{3}}(\tau);\, b^{(\mu)}_{(\infty)\,\check{r}}(\tau)\,)$, with

$l^{(\mu)}_{(\infty)}$ $\tau$-independent, satisfying $b^{(\mu)}_{(\infty)\,A}\, {}^4\eta_{(\mu)(\nu)}\, b^{(\nu)}_{(\infty)\,B} = {}^4\eta_{AB}$ for every $\tau$ [at this level there is no reason to assume that $l^{(\mu)}_{(\infty)}$ is tangent to $S_\infty$, as the normal $l^\mu$ to $\Sigma_\tau$, see Appendix A]. There will be transformation coefficients $b^\mu_A(\tau,\vec{\sigma})$ from the adapted coordinates $\sigma^A = (\tau, \sigma^{\check{r}})$ to coordinates $x^\mu = z^\mu(\sigma^A)$ in an atlas of $M^4$, such that in a chart at spatial infinity one has $z^\mu(\tau,\vec{\sigma}) \rightarrow \delta^\mu_{(\mu)} z^{(\mu)}(\tau,\vec{\sigma})$ and $b^\mu_A(\tau,\vec{\sigma}) \rightarrow \delta^\mu_{(\mu)} b^{(\mu)}_{(\infty)A}(\tau)$ [for $r \rightarrow \infty$ one has ${}^4g_{\mu\nu} \rightarrow \delta^{(\mu)}_\mu \delta^{(\nu)}_\nu {}^4\eta_{(\mu)(\nu)}$ and ${}^4g_{AB} = b^\mu_A\, {}^4g_{\mu\nu} b^\nu_B \rightarrow b^{(\mu)}_{(\infty)A}\, {}^4\eta_{(\mu)(\nu)} b^{(\nu)}_{(\infty)B} = {}^4\eta_{AB}$ ]. In this way one defines the atlas $\mathcal{C}$ of the allowed coordinate systems of $M^4$.

Dirac [8] and, then, Regge and Teitelboim [5] proposed that the asymptotic Minkowski rectangular coordinates $z^{(\mu)}_{(\infty)}(\tau,\vec{\sigma}) = x^{(\mu)}_{(\infty)}(\tau) + b^{(\mu)}_{(\infty)\check{r}}(\tau)\sigma^{\check{r}}$ should define 10 new independent degrees of freedom at the spatial boundary $S_\infty$ [with ten associated conjugate momenta], as it happens for Minkowski parametrized theories [10] when the extra configurational variables $z^{(\mu)}(\tau,\vec{\sigma})$ (describing the embedded spacelike hypersurface and not existing in curved spacetimes) are reduced to 10 degrees of freedom by the restriction to spacelike hyperplanes [defined by $z^{(\mu)}(\tau,\vec{\sigma}) \approx x^{(\mu)}_s(\tau) + b^{(\mu)}_{\check{r}}(\tau)\sigma^{\check{r}}$], but with these 10 degrees of freedom being gauge variables (independence from the choice of the hyperplane) due to 10 surviving first class constraints.

In Minkowski parametrized theories for isolated systems restricted to spacelike hyperplanes (see II) it can be shown [10] that these 20 variables are:

i) $x^{(\mu)}_s(\tau), p^{(\mu)}_s$ [$\{x^{(\mu)}_s, p^{(\nu)}_s\} = -{}^4\eta^{(\mu)(\nu)}$], parametrizing the origin of the coordinates on the family of spacelike hyperplanes. The four constraints $\mathcal{H}^{(\mu)}(\tau) \approx p^{(\mu)}_s - p^{(\mu)}_{sys} \approx 0$ say that $p^{(\mu)}_s$ is determined by the 4-momentum of the isolated system.
ii) $b^{(\mu)}_A(\tau)$ (with the $b^{(\mu)}_r(\tau)$'s being three orthogonal spacelike unit vectors generating the fixed $\tau$-independent timelike unit normal $b^{(\mu)}_\tau = l^{(\mu)}$ to the hyperplanes) and $S^{(\mu)(\nu)}_s = -S^{(\nu)(\mu)}_s$ with the orthonormality constraints $b^{(\mu)}_A\, {}^4\eta_{(\mu)(\nu)} b^{(\nu)}_B = {}^4\eta_{AB}$. The non-vanishing Dirac brackets enforcing the orthonormality constraints [57,10] for the $b^{(\mu)}_A$'s are

$$\{b^{(\rho)}_A, S^{(\mu)(\nu)}_s\} = {}^4\eta^{(\rho)(\mu)} b^{(\nu)}_A - {}^4\eta^{(\rho)(\nu)} b^{(\mu)}_A,$$

$$\{S^{(\mu)(\nu)}_s, S^{(\alpha)(\beta)}_s\} = C^{(\mu)(\nu)(\alpha)(\beta)}_{(\gamma)(\delta)} S^{(\gamma)(\delta)}_s$$

with $C^{(\mu)(\nu)(\alpha)(\beta)}_{(\gamma)(\delta)}$ the structure constants of the Lorentz algebra. Then one has that $p^{(\mu)}_s$, $J^{(\mu)(\nu)}_s = x^{(\mu)}_s p^{(\nu)}_s - x^{(\nu)}_s p^{(\mu)}_s + S^{(\mu)(\nu)}_s$, satisfy the algebra of the Poincaré group, with $S^{(\mu)(\nu)}_s$ playing the role of the spin tensor. The other six constraints $\mathcal{H}^{(\mu)(\nu)}(\tau) \approx S^{(\mu)(\nu)}_s - S^{(\mu)(\nu)}_{sys} \approx 0$ say that $S^{(\mu)(\nu)}_s$ coincides the spin tensor of the isolated system.

Let us remark that, for each configuration of an isolated system there is a privileged family of hyperplanes (the Wigner hyperplanes orthogonal to $p^{(\mu)}_s$, existing when $\epsilon p^2_s > 0$) corresponding to the intrinsic rest-frame of the isolated system. If we choose these hyperplanes with suitable gauge fixings, we remain with only the four constraints $\mathcal{H}^{(\mu)}(\tau) \approx 0$, which can be rewritten as

$$\sqrt{\epsilon p^2_s} \approx [invariant\, mass\, of\, the\, isolated\, system\, under\, investigation] = M_{sys};$$
$$\vec{p}_{sys} = [3 - momentum\, of\, the\, isolated\, system\, inside\, the\, Wigner\, hyperplane] \approx 0.$$

There is no more a restriction on $p_s^{(\mu)}$, because $u_s^{(\mu)}(p_s) = p_s^{(\mu)}/p_s^2$ gives the orientation of the Wigner hyperplanes containing the isolated system with respect to an arbitrary given external observer.

In this special gauge we have $b^{(\mu)}_A \equiv L^{(\mu)}{}_A(p_s, \overset{\circ}{p}_s)$ (the standard Wigner boost for timelike Poincaré orbits), $S_s^{(\mu)(\nu)} \equiv S_{sys}^{(\mu)(\nu)}$, $\tilde{\lambda}_{(\mu)(\nu)}(\tau) \equiv 0$, and the only remaining canonical variables are the non-covariant Newton-Wigner-like canonical "external" center-of-mass coordinate $\tilde{x}_s^{(\mu)}(\tau)$ (living on the Wigner hyperplanes) and $p_s^{(\mu)}$. Now 3 degrees of freedom of the isolated system [an "internal" center-of-mass 3-variable $\vec{\sigma}_{sys}$ defined inside the Wigner hyperplane and conjugate to $\vec{p}_{sys}$] become gauge variables [the natural gauge fixing is $\vec{\sigma}_{sys} \approx 0$, so that it coincides with the origin $x_s^{(\mu)}(\tau) = z^{(\mu)}(\tau, \vec{\sigma} = 0)$ of the Wigner hyperplane], while the $\tilde{x}^{(\mu)}(\tau)$ is playing the role of a kinematical external center of mass for the isolated system and may be interpreted as a decoupled observer with his parametrized clock (point particle clock). All the fields living on the Wigner hyperplane are now either Lorentz scalar or with their 3-indices transforming under Wigner rotations (induced by Lorentz transformations in Minkowski spacetime) as any Wigner spin 1 index. Let us remark that the constant $x_s^{(\mu)}(0)$ [and, therefore, also $\tilde{x}_s^{(\mu)}(0)$] is arbitrary, reflecting the arbitrariness in the absolute location of the origin of the "internal" coordinates on each hyperplane in Minkowski spacetime.

One obtains in this way a new kind of instant form of the dynamics, the "Wigner-covariant 1-time rest-frame instant form" [10] with a universal breaking of Lorentz covariance. It is the special relativistic generalization of the non-relativistic separation of the center of mass from the relative motion [$H = \frac{\vec{P}^2}{2M} + H_{rel}$]. The role of the center of mass is taken by the Wigner hyperplane, identified by the point $\tilde{x}^{(\mu)}(\tau)$ and by its normal $p_s^{(\mu)}$. The invariant mass $M_{sys}$ of the system replaces the non-relativistic Hamiltonian $H_{rel}$ for the relative degrees of freedom, after the addition of the gauge-fixing $T_s - \tau \approx 0$ [identifying the time parameter $\tau$, labelling the leaves of the foliation, with the Lorentz scalar time of the center of mass in the rest frame, $T_s = p_s \cdot \tilde{x}_s/M_{sys}$; $M_{sys}$ generates the evolution in this time].

The determination of $\vec{\sigma}_{sys}$ may be done with the group theoretical methods of Ref. [58]: given a realization on the phase space of a given system of the ten Poincaré generators one can build three 3-position variables only in terms of them, which in our case of a system on the Wigner hyperplane with $\vec{p}_{sys} \approx 0$ are: i) a canonical center of mass (the "internal" center of mass $\vec{\sigma}_{sys}$); ii) a non-canonical Møller center of energy $\vec{\sigma}_{sys}^{(E)}$; iii) a non-canonical Fokker-Price center of inertia $\vec{\sigma}_{sys}^{(FP)}$. Due to $\vec{p}_{sys} \approx 0$, we have $\vec{\sigma}_{sys} \approx \vec{\sigma}_{sys}^{(FP)} \approx \vec{\sigma}_{sys}^{(E)} = \{boostgenerator/energy\}$. By adding the gauge fixings $\vec{\sigma}_{sys} \approx 0$ one can show that the origin $x_s^{(\mu)}(\tau)$ becomes simultaneously the Dixon center of mass of an extended object and both the Pirani and Tulczyjew centroids (see Ref. [59] for the application of these methods to find the center of mass of a configuration of the Klein-Gordon field after the preliminary work of Ref. [60]). With similar methods one can construct three "external" collective positions (all located on the Wigner hyperplane): i) the "external" canonical non-covariant center of mass $\tilde{x}_s^{(\mu)}$; ii) the "external" non-canonical and non-covariant Møller center of energy $R_s^{(\mu)}$; iii) the "external" covariant non-canonical Fokker-Price center of inertia $Y_s^{(\mu)}$ (when there are the gauge fixings $\vec{\sigma}_{sys} \approx 0$ it also coincides with the origin $x_s^{(\mu)}$). It turns out that the Wigner hyperplane is the natural setting for the study of the Dixon multipoles of extended relativistic systems [61] and for defining the canonical relative variables with respect to the

center of mass. The Wigner hyperplane with its natural Euclidean metric structure offers a natural solution to the problem of boost for lattice gauge theories and realizes explicitly the Machian aspect of dynamics that only relative motions are relevant.

Analogously, in Dirac's approach to metric gravity one expects that the 20 extra variables of the Dirac proposal should be replaced by a set of this kind: $x^{(\mu)}_{(\infty)}(\tau)$, $p^{(\mu)}_{(\infty)}$, $b^{(\mu)}_{(\infty)A}(\tau)$ [with $b^{(\mu)}_{(\infty)\tau} = l^{(\mu)}_{(\infty)}$ $\tau$-independent and coinciding with the asymptotic normal to $\Sigma_\tau$, tangent to $S_\infty$], $S^{(\mu)(\nu)}_{(\infty)}$, with the previous Dirac brackets implying the orthonormality constraints $b^{(\mu)}_{(\infty)A}\, {}^4\eta_{(\mu)(\nu)} b^{(\nu)}_{(\infty)B} = {}^4\eta_{AB}$. Moreover, $p^{(\mu)}_{(\infty)}$ and $J^{(\mu)(\nu)}_{(\infty)} = x^{(\mu)}_{(\infty)} p^{(\nu)}_{(\infty)} - x^{(\nu)}_{(\infty)} p^{(\mu)}_{(\infty)} + S^{(\mu)(\nu)}_{(\infty)}$ should satisfy a Poincaré algebra. In analogy with Minkowski parametrized theories restricted to spacelike hyperplanes, one expects to have 10 extra first class constraints of the type

$$\begin{aligned}
&p^{(\mu)}_{(\infty)} - P^{(\mu)}_{ADM} \approx 0,\\
&S^{(\mu)(\nu)}_{(\infty)} - S^{(\mu)(\nu)}_{ADM} \approx 0
\end{aligned}$$

with $P^{(\mu)}_{ADM}$, $S^{(\mu)(\nu)}_{ADM}$ related to the ADM Poincaré charges and 10 extra Dirac multipliers $\tilde{\lambda}_{(\mu)}(\tau)$, $\tilde{\lambda}_{(\mu)(\nu)}(\tau)$, in front of them in the Dirac Hamiltonian. The origin $x^{(\mu)}_{(\infty)}$ is going to play the role of an "external" decoupled observer with his parametrized clock. The main problem with respect to Minkowski parametrized theory on spacelike hyperplanes is that it is not known which could be the ADM spin part $S^{(\mu)(\nu)}_{ADM}$ of the ADM Lorentz charge $J^{(\mu)(\nu)}_{ADM}$.

The way out from these problems is based on the following observation. If we replace $p^{(\mu)}_{(\infty)}$ and $S^{(\mu)(\nu)}_{(\infty)}$, whose Poisson algebra is the direct sum of an Abelian algebra of translations and of a Lorentz algebra, with the new variables (with indices adapted to $\Sigma_\tau$)

$$\begin{aligned}
&p^A_{(\infty)} = b^A_{(\infty)(\mu)} p^{(\mu)}_{(\infty)},\\
&J^{AB}_{(\infty)} \overset{def}{=} b^A_{(\infty)(\mu)} b^B_{(\infty)(\nu)} S^{(\mu)(\nu)}_{(\infty)} \ [\neq b^A_{(\infty)(\mu)} b^B_{(\infty)(\nu)} J^{(\mu)(\nu)}_{(\infty)}],
\end{aligned}$$

the Poisson brackets for $p^{(\mu)}_{(\infty)}$, $b^{(\mu)}_{(\infty)A}$, $S^{(\mu)(\nu)}_{(\infty)}$ [one has $\{b^A_{(\infty)(\gamma)}, S^{(\nu)(\rho)}_{(\infty)}\} = \eta^{(\nu)}_{(\gamma)} b^{A(\rho)}_{(\infty)} - \eta^{(\rho)}_{(\gamma)} b^{A(\nu)}_{(\infty)}$], imply

$$\begin{aligned}
&\{p^A_{(\infty)}, p^B_{(\infty)}\} = 0,\\
&\{p^A_{(\infty)}, J^{BC}_{(\infty)}\} = {}^4g^{AC}_{(\infty)} p^B_{(\infty)} - {}^4g^{AB}_{(\infty)} p^C_{(\infty)},\\
&\{J^{AB}_{(\infty)}, J^{CD}_{(\infty)}\} = -(\delta^B_E \delta^C_F\, {}^4g^{AD}_{(\infty)} + \delta^A_E \delta^D_F\, {}^4g^{BC}_{(\infty)} - \delta^B_E \delta^D_F\, {}^4g^{AC}_{(\infty)} - \delta^A_E \delta^C_F\, {}^4g^{BD}_{(\infty)}) J^{EF}_{(\infty)} =\\
&= -C^{ABCD}_{EF} J^{EF}_{(\infty)},
\end{aligned} \tag{9}$$

where ${}^4g^{AB}_{(\infty)} = b^A_{(\infty)(\mu)}\, {}^4\eta^{(\mu)(\nu)} b^B_{(\infty)(\nu)} = {}^4\eta^{AB}$ since the $b^{(\mu)}_{(\infty)A}$ are flat tetrad in both kinds of indices. Therefore, we get the algebra of a realization of the Poincaré group [this explains the notation $J^{AB}_{(\infty)}$] with all the structure constants inverted in the sign (transition from a left to a right action).

This implies that the Poincaré generators $P^A_{ADM}$, $J^{AB}_{ADM}$ in $\Sigma_\tau$-adapted coordinates should define in the asymptotic Dirac rectangular coordinates a momentum $P^{(\mu)}_{ADM} = b^{(\mu)}_A P^A_{ADM}$ and only an ADM spin tensor $S^{(\mu)(\nu)}_{ADM}$ [to define an angular momentum tensor $J^{(\mu)(\nu)}_{ADM}$ one should find an "external center of mass of the gravitational field" $X^{(\mu)}_{ADM}[{}^3g, {}^3\tilde{\Pi}]$ (see Ref. [60,59] for

the Klein-Gordon case) conjugate to $P^{(\mu)}_{ADM}$, so that $J^{(\mu)(\nu)}_{ADM} = X^{(\mu)}_{ADM} P^{(\nu)}_{ADM} - X^{(\nu)}_{ADM} P^{(\mu)}_{ADM} + S^{(\mu)(\nu)}_{ADM}$].

Therefore we shall assume the existence of a global coordinate system $\{\sigma^{\check r}\}$ on $\Sigma_\tau$ , in which we have

$$\begin{aligned}
N(\tau,\vec\sigma) &= N_{(as)}(\tau,\vec\sigma) + m(\tau,\vec\sigma),\\
N_{\check r}(\tau,\vec\sigma) &= N_{(as)\check r}(\tau,\vec\sigma) + m_{\check r}(\tau,\vec\sigma),
\end{aligned}$$

$$\begin{aligned}
N_{(as)}(\tau,\vec\sigma) &= -\tilde\lambda_{(\mu)}(\tau) l^{(\mu)}_{(\infty)} - l^{(\mu)}_{(\infty)} \tilde\lambda_{(\mu)(\nu)}(\tau) b^{(\nu)}_{(\infty)\check s}(\tau)\sigma^{\check s} =\\
&= -\tilde\lambda_\tau(\tau) - \frac{1}{2}\tilde\lambda_{\tau\check s}(\tau)\sigma^{\check s},\\
N_{(as)\check r}(\tau,\vec\sigma) &= -b^{(\mu)}_{(\infty)\check r}(\tau)\tilde\lambda_{(\mu)}(\tau) - b^{(\mu)}_{(\infty)\check r}(\tau)\tilde\lambda_{(\mu)(\nu)}(\tau) b^{(\nu)}_{(\infty)\check s}(\tau)\sigma^{\check s} =\\
&= -\tilde\lambda_{\check r}(\tau) - \frac{1}{2}\tilde\lambda_{\check r\check s}(\tau)\sigma^{\check s},
\end{aligned} \tag{10}$$

with $m(\tau,\vec\sigma)$, $m_{\check r}(\tau,\vec\sigma)$, given by Eqs.(4) [they still contain odd supertranslations].

Let us remark that the quoted restriction to constant improper gauge transformations [$\alpha^{(rigid)} = const.$] of Yang-Mills theory is here replaced by the assumed forms of $N_{(as)}(\tau,\vec\sigma)$, $N_{(as)\check r}(\tau,\vec\sigma)$.

This very strong assumption implies that one is selecting asymptotically at spatial infinity only coordinate systems in which the lapse and shift functions have behaviours similar to those of Minkowski spacelike hyperplanes, so that the allowed foliations of the 3+1 splittings of the spacetime $M^4$ are restricted to have the leaves $\Sigma_\tau$ approaching these Minkowski hyperplanes at spatial infinity in a way independent from the direction if supertranslations are absent. But this is coherent with Dirac's choice of asymptotic rectangular coordinates [modulo 3-diffeomorphisms not changing the nature of the coordinates] and with the assumptions used to define the asymptotic Poincaré charges. It is also needed to eliminate coordinate transformations not becoming the identity at spatial infinity [they are not associated with the gravitational fields of isolated systems [62]].

By using $\tilde\lambda_A(\tau) = \{\tilde\lambda_\tau(\tau); \tilde\lambda_{\check r}(\tau)\}$, $\tilde\lambda_{AB}(\tau) = -\tilde\lambda_{BA}(\tau)$, $n(\tau,\vec\sigma)$, $n_{\check r}(\tau,\vec\sigma)$ as new configuration variables [replacing $N(\tau,\vec\sigma)$ and $N_{\check r}(\tau,\vec\sigma)$] in the ADM Lagrangian (see Section V of I and Appendix A) only produces the replacement of the first class constraints

$\tilde\pi^N(\tau,\vec\sigma) \approx 0$, $\tilde\pi^{\check r}_{\vec N}(\tau,\vec\sigma) \approx 0$,

with the new first class constraints

$\tilde\pi^n(\tau,\vec\sigma) \approx 0$, $\tilde\pi^{\check r}_{\vec n}(\tau,\vec\sigma) \approx 0$, $\tilde\pi^A(\tau) \approx 0$, $\tilde\pi^{AB}(\tau) = -\tilde\pi^{BA}(\tau) \approx 0$,

corresponding to the vanishing of the canonical momenta conjugate to the new configuration variables [we assume the Poisson brackets $\{\tilde\lambda_A(\tau), \tilde\pi^B(\tau)\} = \delta^B_A$, $\{\tilde\lambda_{AB}(\tau), \tilde\pi^{CD}(\tau)\} = \delta^C_A\delta^D_B - \delta^D_A\delta^C_B$]. The only change in the Dirac Hamiltonian $H_{(D)ADM}$ of Eq.(A2) is

$\int d^3\sigma [\lambda_N \tilde\pi^N + \lambda^{\vec N}_{\check r} \tilde\pi^{\check r}_{\vec N}](\tau,\vec\sigma) \mapsto \zeta_A(\tau)\tilde\pi^A(\tau) + \zeta_{AB}(\tau)\tilde\pi^{AB}(\tau) + \int d^3\sigma [\lambda_n \tilde\pi^n + \lambda^{\vec n}_{\check r}\tilde\pi^{\check r}_{\vec n}](\tau,\vec\sigma)$

with $\zeta_A$, $\zeta_{AB}$ Dirac's multipliers.

It seems impossible to have a reformulation of metric gravity corresponding to the fully parametrized Minkowski theory on arbitrary spacelike hypersurfaces, first because on them there is not a unique definition of lapse and shift functions (see II) and second because the coefficients $b^\mu_A$ are not tetrads in curved spacetimes like the $b^{(\mu)}_A = z^{(\mu)}_A$ in Minkowski spacetime. The existence of this holonomic basis for vector fields on it allows to use the coordinates $z^{(\mu)}(\tau,\vec{\sigma})$ as configurational variables in parametrized Minkowski theories. Instead in the ADM theory the configuration variables are only $N$, $N_{\check{r}}$, ${}^3g_{\check{r}\check{s}}$, because the $b^\mu_A$ are now a non-holonomic basis.

Deferring to the next Section a discussion about the privileged observers associated with these asymptotic Minkowskian hyperplanes, let us come back to the definition of the asymptotic Poincaré charges in metric gravity. The splitting (10) of the lapse and shift functions and the addition to the canonical Hamiltonian $H_{(c)ADM}$ of an appropriate surface integral $H_\infty$, defined in Refs. [5,6], produce a modified canonical Hamiltonian which is differentiable and finite: namely its variation is linear in $\delta\, {}^3g_{\check{r}\check{s}}$, $\delta\, {}^3\tilde{\Pi}^{\check{r}\check{s}}$. In our notation, by using the surface integral of Eqs.(A3) and $N_{(as)}(\tau,\vec{\sigma})$ and $N_{(as)\check{r}}(\tau,\vec{\sigma})$ of Eqs.(10), the modified canonical Hamiltonian is [we use the global coordinate system $\{\sigma^{\check{r}}\}$ of Eq.(10); we remember that $k = c^3/16\pi G$ with G the Newton constant]

$$\begin{aligned}
\hat{H}_{(c)ADM} &= \int d^3\sigma [N\tilde{\mathcal{H}} + N_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) = \\
&= \int d^3\sigma [(N_{(as)} + m)\tilde{\mathcal{H}} + (N_{(as)\check{r}} + m_{\check{r}})\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) \mapsto \\
\mapsto \hat{H}^{'}_{(c)ADM} &= \hat{H}^{'}_{(c)ADM}[N, N^{\check{r}}] = \hat{H}_{(c)ADM} + H_\infty = \\
&= \int d^3\sigma [N\tilde{\mathcal{H}} + N_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) - \\
&- \int_{S^2_{\tau,\infty}} d^2\Sigma_{\check{u}} \{\epsilon k \sqrt{\gamma}\, {}^3g^{\check{u}\check{v}}\, {}^3g^{\check{r}\check{s}} [N(\partial_{\check{r}}\, {}^3g_{\check{v}\check{s}} - \partial_{\check{v}}\, {}^3g_{\check{r}\check{s}}) + \\
&+ \partial_{\check{u}} N({}^3g_{\check{r}\check{s}} - \delta_{\check{r}\check{s}}) - \partial_{\check{r}} N({}^3g_{\check{s}\check{v}} - \delta_{\check{s}\check{v}})] - 2N_{\check{r}}\, {}^3\tilde{\Pi}^{\check{r}\check{u}}\}(\tau,\vec{\sigma}) = \\
&= \int d^3\sigma [(N_{(as)} + m)\tilde{\mathcal{H}} + (N_{(as)\check{r}} + m_{\check{r}})\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) - \\
&- \int_{S^2_{\tau,\infty}} d^2\Sigma_{\check{u}} \{\epsilon k \sqrt{\gamma}\, {}^3g^{\check{u}\check{v}}\, {}^3g^{\check{r}\check{s}} [N_{(as)}(\partial_{\check{r}}\, {}^3g_{\check{v}\check{s}} - \partial_{\check{v}}\, {}^3g_{\check{r}\check{s}}) + \\
&+ \partial_{\check{u}} N_{(as)}({}^3g_{\check{r}\check{s}} - \delta_{\check{r}\check{s}}) - \partial_{\check{r}} N_{(as)}({}^3g_{\check{s}\check{v}} - \delta_{\check{s}\check{v}})] - 2N_{(as)\check{r}}\, {}^3\tilde{\Pi}^{\check{r}\check{u}}\}(\tau,\vec{\sigma}) = \\
&= \int d^3\sigma [(N_{(as)} + m)\tilde{\mathcal{H}} + (N_{(as)\check{r}} + m_{\check{r}})\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) + \\
&+ \tilde{\lambda}_{(\mu)}(\tau) P^{(\mu)}_{ADM} + \tilde{\lambda}_{(\mu)(\nu)}(\tau) S^{(\mu)(\nu)}_{ADM} = \\
&= \int d^3\sigma [(N_{(as)} + m)\tilde{\mathcal{H}} + (N_{(as)\check{r}} + m_{\check{r}})\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) + \\
&+ \tilde{\lambda}_A(\tau) P^A_{ADM} + \frac{1}{2}\tilde{\lambda}_{AB}(\tau) J^{AB}_{ADM} \approx \\
&\approx \tilde{\lambda}_A(\tau) P^A_{ADM} + \frac{1}{2}\tilde{\lambda}_{AB}(\tau) J^{AB}_{ADM}.
\end{aligned} \tag{11}$$

Indeed, by putting $N = N_{(as)}$, $N_{\check{r}} = N_{(as)\check{r}}$ in the surface integrals, the added term (A3) becomes the given linear combination of the strong ADM Poincaré charges $P^A_{ADM}$, $J^{AB}_{ADM}$

[5,6] first identified in the linearized theory [4] [they are called "strong conserved improper charges" in analogy with the strong (surface integrals) Yang-Mills non Abelian charges, whose conservation is not a consequence of the equations of motion [40] in that case]:

$$
\begin{aligned}
P^{\tau}_{ADM} &= \epsilon k \int_{S^2_{\tau,\infty}} d^2\Sigma_{\check u}[\sqrt{\gamma}\; {}^3g^{\check u\check v}\, {}^3g^{\check r\check s}(\partial_{\check r}\, {}^3g_{\check v\check s} - \partial_{\check v}\, {}^3g_{\check r\check s})](\tau,\vec\sigma),\\
P^{\check r}_{ADM} &= -2\int_{S^2_{\tau,\infty}} d^2\Sigma_{\check u}\, {}^3\tilde\Pi^{\check r\check u}(\tau,\vec\sigma),\\
J^{\tau\check r}_{ADM} &= \epsilon k \int_{S^2_{\tau,\infty}} d^2\Sigma_{\check u}\sqrt{\gamma}\; {}^3g^{\check u\check v}\, {}^3g^{\check n\check s}\cdot\\
&\quad\cdot\ [\sigma^{\check r}(\partial_{\check n}\, {}^3g_{\check v\check s} - \partial_{\check v}\, {}^3g_{\check n\check s}) + \delta^{\check r}_{\check v}({}^3g_{\check n\check s} - \delta_{\check n\check s}) - \delta^{\check r}_{\check n}({}^3g_{\check s\check v} - \delta_{\check s\check v})](\tau,\vec\sigma),\\
J^{\check r\check s}_{ADM} &= \int_{S^2_{\tau,\infty}} d^2\Sigma_{\check u}[\sigma^{\check r}\, {}^3\tilde\Pi^{\check s\check u} - \sigma^{\check s}\, {}^3\tilde\Pi^{\check r\check u}](\tau,\vec\sigma),\\
&\\
P^{(\mu)}_{ADM} &= l^{(\mu)}P^{\tau}_{ADM} + b^{(\mu)}_{(\infty)\check r}(\tau)P^{\check r}_{ADM} = b^{(\mu)}_{(\infty)A}(\tau)P^A_{ADM},\\
S^{(\mu)(\nu)}_{ADM} &= [l^{(\mu)}_{(\infty)}b^{(\nu)}_{(\infty)\check r}(\tau) - l^{(\nu)}_{(\infty)}b^{(\mu)}_{(\infty)\check r}(\tau)]J^{\tau\check r}_{ADM} +\\
&+ [b^{(\mu)}_{(\infty)\check r}(\tau)b^{(\nu)}_{(\infty)\check s}(\tau) - b^{(\nu)}_{(\infty)\check r}(\tau)b^{(\mu)}_{(\infty)\check s}(\tau)]J^{\check r\check s}_{ADM} =\\
&= [b^{(\mu)}_{(\infty)A}(\tau)b^{(\nu)}_{(\infty)B}(\tau) - b^{(\nu)}_{(\infty)A}(\tau)b^{(\mu)}_{(\infty)B}(\tau)]J^{AB}_{ADM},\\
&\\
&\tilde\lambda_A(\tau) = \tilde\lambda_{(\mu)}(\tau)b^{(\mu)}_{(\infty)A}(\tau),\qquad \tilde\lambda_{(\mu)}(\tau) = b^A_{(\infty)(\mu)}(\tau)\tilde\lambda_A(\tau),\\
&\tilde\lambda_{AB}(\tau) = \tilde\lambda_{(\mu)(\nu)}(\tau)[b^{(\mu)}_{(\infty)A}b^{(\nu)}_{(\infty)B} - b^{(\nu)}_{(\infty)A}b^{(\mu)}_{(\infty)B}](\tau) = 2[\tilde\lambda_{(\mu)(\nu)}b^{(\mu)}_{(\infty)A}b^{(\nu)}_{(\infty)B}](\tau),\\
&\tilde\lambda_{(\mu)(\nu)}(\tau) = \frac{1}{4}[b^A_{(\infty)(\mu)}b^B_{(\infty)(\nu)} - b^B_{(\infty)(\mu)}b^A_{(\infty)(\nu)}](\tau)\tilde\lambda_{AB}(\tau) =\\
&= \frac{1}{2}[b^A_{(\infty)(\mu)}b^B_{(\infty)(\nu)}\tilde\lambda_{AB}](\tau).
\end{aligned}
\tag{12}
$$

Here $J^{\tau\check r}_{ADM} = -J^{\check r\tau}_{ADM}$ by definition and the inverse asymptotic tetrads are defined by $b^A_{(\infty)(\mu)}b^{(\nu)}_{(\infty)B} = \delta^A_B$, $b^A_{(\infty)(\mu)}b^{(\nu)}_{(\infty)A} = \delta^{(\nu)}_{(\mu)}$.

As shown in Ref. [5,6], the parity conditions of Eqs.(2) are necessary to have a well defined and finite 3-angular-momentum $J^{\check r\check s}_{ADM}$: in Appendix B of Ref. [6] there is an explicit example of initial data satisfying the constraints but not the parity conditions , for which the 3-angular-momentum is infinite [moreover, it is shown that the conditions of the SPI formalism to kill supertranslations and pick out a unique asymptotic Poincaré group (the vanishing of the first-order asymptotic part of the pseudomagnetic Weyl tensor) may give infinite 3-angular-momentum if the parity conditions are not added].

The definition of the boosts $J^{\tau\check r}_{ADM}$ given in Ref. [6] is not only differentiable like the one in Ref. [5], but also finite. As seen in Section II, the problem of boosts is still open. However, for any isolated system the boost part of the conserved Poincaré group cannot be an independent variable [only the Poincaré Casimirs (giving the invariant mass and spin of the system) are relevant and not the Casimirs of the Lorentz subgroup]. At the end of this Section this point will be clarified by giving the explicit realization of the Poincaré generators in the rest-frame instant form (they are independent from the ADM boosts).

Let us use Eqs.(A5) with $N = N_{(as)}$, $N_{\check r} = N_{(as)\check r}$, and Eqs.(10) to rewrite Eq.(11) in the following form

$$
\begin{aligned}
\hat{H}^{'}_{(c)ADM} &= \int d^3\sigma [m\tilde{\mathcal{H}} + m_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) + \\
&+ \tilde{\lambda}_\tau(\tau)[-\int d^3\sigma \tilde{\mathcal{H}}(\tau,\vec{\sigma}) + P^\tau_{ADM}] + \tilde{\lambda}_{\check{r}}(\tau)[-\int d^3\sigma\, {}^3\tilde{\mathcal{H}}^{\check{r}}(\tau,\vec{\sigma}) + P^{\check{r}}_{ADM}] + \\
&+ \tilde{\lambda}_{\tau\check{r}}(\tau)[-\frac{1}{2}\int d^3\sigma \sigma^{\check{r}}\, \tilde{\mathcal{H}}(\tau,\vec{\sigma}) + J^{\tau\check{r}}_{ADM}] + \\
&+ \tilde{\lambda}_{\check{r}\check{s}}(\tau)[-\frac{1}{2}\int d^3\sigma \sigma^{\check{s}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}(\tau,\vec{\sigma}) + J^{\check{r}\check{s}}_{ADM}] = \\
&= \int d^3\sigma [m\tilde{\mathcal{H}} + m_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) + \\
&+ \int d^3\sigma \{\epsilon N_{(as)}[k\sqrt{\gamma}\, {}^3g^{\check{r}\check{s}}({}^3\Gamma^{\check{u}}_{\check{r}\check{v}}\, {}^3\Gamma^{\check{v}}_{\check{s}\check{u}} - {}^3\Gamma^{\check{u}}_{\check{r}\check{s}}\, {}^3\Gamma^{\check{v}}_{\check{v}\check{u}}) - \\
&- \frac{1}{2k\sqrt{\gamma}}{}^3G_{\check{r}\check{s}\check{u}\check{v}}\, {}^3\tilde{\Pi}^{\check{r}\check{s}}\, {}^3\tilde{\Pi}^{\check{u}\check{v}}] + \\
&+ \epsilon k({}^3g_{\check{v}\check{s}} - \delta_{\check{v}\check{s}})\partial_{\check{r}}[\sqrt{\gamma}\partial_{\check{u}}N_{(as)}({}^3g^{\check{r}\check{s}}\, {}^3g^{\check{u}\check{v}} - {}^3g^{\check{u}\check{r}}\, {}^3g^{\check{v}\check{s}})] - \\
&- 2N_{(as)\check{r}}\, {}^3\Gamma^{\check{r}}_{\check{s}\check{u}}\, {}^3\tilde{\Pi}^{\check{s}\check{u}} + 2\partial_{\check{u}}N_{(as)\check{r}}\, {}^3\tilde{\Pi}^{\check{r}\check{u}}\}(\tau,\vec{\sigma}) = \\
&= \int d^3\sigma [m\tilde{\mathcal{H}} + m_{\check{r}}{}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) + \tilde{\lambda}_{(\mu)}(\tau)\hat{P}^{(\mu)}_{ADM} + \tilde{\lambda}_{(\mu)(\nu)}(\tau)\hat{S}^{(\mu)(\nu)}_{ADM} = \\
&= \int d^3\sigma [m\tilde{\mathcal{H}} + m_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau,\vec{\sigma}) + \tilde{\lambda}_A(\tau)\hat{P}^A_{ADM} + \frac{1}{2}\tilde{\lambda}_{AB}(\tau)\hat{J}^{AB}_{ADM} \approx \\
&\approx \tilde{\lambda}_A(\tau)\hat{P}^A_{ADM} + \frac{1}{2}\tilde{\lambda}_{AB}(\tau)\hat{J}^{AB}_{ADM}, \\
\hat{H}^{'}_{(D)ADM} &= \hat{H}^{'}_{(c)ADM}[m, m^{\check{r}}] + \\
&+ \int d^3\sigma [\lambda_n \tilde{\pi}^n + \lambda^{\vec{n}}_r \tilde{\pi}^r_{\vec{n}}](\tau,\vec{\sigma}) + \zeta_A(\tau)\tilde{\pi}^A(\tau) + \zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau),
\end{aligned} \tag{13}
$$

with the following "weak conserved improper charges" $\hat{P}^A_{ADM}$, $\hat{J}^{AB}_{ADM}$ [these volume expressions (the analogue of the weak Yang-Mills non Abelian charges) for the ADM 4-momentum are used in Ref. [63] in the study of the positiviteness of the energy; the weak charges are Noether charges]

$$
\begin{aligned}
\hat{P}^\tau_{ADM} &= \int d^3\sigma \epsilon [k\sqrt{\gamma}\; {}^3g^{\check{r}\check{s}}({}^3\Gamma^{\check{u}}_{\check{r}\check{v}}\, {}^3\Gamma^{\check{v}}_{\check{s}\check{u}} - {}^3\Gamma^{\check{u}}_{\check{r}\check{s}}\, {}^3\Gamma^{\check{v}}_{\check{v}\check{u}}) - \\
&- \frac{1}{2k\sqrt{\gamma}}{}^3G_{\check{r}\check{s}\check{u}\check{v}}\, {}^3\tilde{\Pi}^{\check{r}\check{s}}\, {}^3\tilde{\Pi}^{\check{u}\check{v}}](\tau,\vec{\sigma}), \\
\hat{P}^{\check{r}}_{ADM} &= -2\int d^3\sigma\, {}^3\Gamma^{\check{r}}_{\check{s}\check{u}}(\tau,\vec{\sigma})\, {}^3\tilde{\Pi}^{\check{s}\check{u}}(\tau,\vec{\sigma}), \\
\hat{J}^{\tau\check{r}}_{ADM} &= -\hat{J}^{\check{r}\tau}_{ADM} = \int d^3\sigma \epsilon \{\sigma^{\check{r}} \\
&\quad [k\sqrt{\gamma}\; {}^3g^{\check{n}\check{s}}({}^3\Gamma^{\check{u}}_{\check{n}\check{v}}\, {}^3\Gamma^{\check{v}}_{\check{s}\check{u}} - {}^3\Gamma^{\check{u}}_{\check{n}\check{s}}\, {}^3\Gamma^{\check{v}}_{\check{v}\check{u}}) - \frac{1}{2k\sqrt{\gamma}}{}^3G_{\check{n}\check{s}\check{u}\check{v}}\, {}^3\tilde{\Pi}^{\check{n}\check{s}}\, {}^3\tilde{\Pi}^{\check{u}\check{v}}] + \\
&+ k\delta^{\check{r}}_{\check{u}}({}^3g_{\check{v}\check{s}} - \delta_{\check{v}\check{s}})\partial_{\check{n}}[\sqrt{\gamma}({}^3g^{\check{n}\check{s}}\, {}^3g^{\check{u}\check{v}} - {}^3g^{\check{n}\check{u}}\, {}^3g^{\check{s}\check{v}})]\}(\tau,\vec{\sigma}), \\
\hat{J}^{\check{r}\check{s}}_{ADM} &= \int d^3\sigma [(\sigma^{\check{r}}\, {}^3\Gamma^{\check{s}}_{\check{u}\check{v}} - \sigma^{\check{s}}\, {}^3\Gamma^{\check{r}}_{\check{u}\check{v}})\, {}^3\tilde{\Pi}^{\check{u}\check{v}}](\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
\hat{P}^{(\mu)}_{ADM} &= l^{(\mu)}_{(\infty)}\hat{P}^\tau_{ADM} + b^{(\mu)}_{(\infty)\check{r}}(\tau)\hat{P}^{\check{r}}_{ADM} = b^{(\mu)}_{(\infty)A}(\tau)\hat{P}^A_{ADM}, \\
\hat{S}^{(\mu)(\nu)}_{ADM} &= [l^{(\mu)}_{(\infty)}b^{(\nu)}_{(\infty)\check{r}}(\tau) - l^{(\nu)}_{(\infty)}b^{(\mu)}_{(\infty)\check{r}}(\tau)]\hat{J}^{\tau\check{r}}_{ADM} +
\end{aligned}
$$

$$
\begin{aligned}
&+ [b^{(\mu )}_{(\infty )\check r}(\tau ) b^{(\nu )}_{(\infty )\check s}(\tau ) - b^{(\nu )}_{(\infty )\check r}(\tau ) b^{(\mu )}_{(\infty )\check s}(\tau )] \hat J^{\check r\check s}_{ADM} =\\
&= [b^{(\mu )}_{(\infty )A} b^{(\nu )}_{(\infty )B} - b^{(\nu )}_{(\infty )A} b^{(\mu )}_{(\infty )B}](\tau ) \hat J^{AB}_{ADM}.
\end{aligned} \tag{14}
$$

In both Refs. [5,6] it is shown that the canonical Hamiltonian $\hat H^{'}_{(c)ADM}[N, N^{\check r}]$ of Eq.(11) with arbitrary $N$, $N^{\check r}$, has the same Poisson brackets as in Eq.(5) for $N = m$, $N^{\check r} = m^{\check r}$ ["proper" gauge transformations]

$$
\{\hat H^{'}_{(c)ADM}[N_1, N_1^{\check r}], \hat H^{'}_{(c)ADM}[N_2, N_2^{\check r}]\} = \hat H^{'}_{(c)ADM}[N_3, N_3^{\check r}],
$$

$$
\begin{aligned}
N_3 &= N_2^{\check r} \partial_{\check r} N_1 - N_1^{\check r} \partial_{\check r} N_2,\\
N_3^{\check r} &= {\cal L}_{\vec N_2} N_1^{\check r} + N_2 \partial^{\check r} N_1 - N_1 \partial^{\check r} N_2 =\\
&= - N_1^{\check s} \partial_{\check s} N_2^{\check r} + N_2^{\check s} \partial_{\check s} N_1^{\check r} + N_2 \partial^{\check r} N_1 - N_1 \partial^{\check r} N_2,
\end{aligned}
$$

$$
\begin{aligned}
if \quad & N_i(\tau ,\vec \sigma ) = m_i(\tau ,\vec \sigma ) - \tilde \lambda_{i\tau}(\tau ) - \frac{1}{2} \tilde \lambda_{i\tau \check u}(\tau ) \sigma^{\check u}, \quad i=1,2,\\
and \quad & N_{i\check r}(\tau ,\vec \sigma ) = m_{i\check r}(\tau ,\vec \sigma ) - \tilde \lambda_{i\check r}(\tau ) - \frac{1}{2} \tilde \lambda_{i\check r\check u}(\tau ) \sigma^{\check u}, \quad i=1,2,\\
& \Downarrow\\
N_3 &= m_3 - \tilde \lambda_{3\tau} - \frac{1}{2} \tilde \lambda_{3\tau \check u} \sigma^{\check u},\\
N_3^{\check r} &= - \epsilon\, {}^3g^{\check r\check s} [m_{3\check s} - \tilde \lambda_{3\check s} - \frac{1}{2} \tilde \lambda_{3\check s\check u} \sigma^{\check u}],\\
&\\
& with
\end{aligned}
$$

$$
\begin{aligned}
\tilde \lambda_{3\tau} &= - \frac{\epsilon}{2} \delta^{\check r\check s} [\tilde \lambda_{1\check r} \tilde \lambda_{2\tau \check s} - \tilde \lambda_{2\check r} \tilde \lambda_{1\tau \check s}],\\
\tilde \lambda_{3\tau \check u} &= - \frac{\epsilon}{2} \delta^{\check r\check s} [\tilde \lambda_{1\check r\check u} \tilde \lambda_{2\tau \check s} - \tilde \lambda_{2\check r\check u} \tilde \lambda_{1\tau \check s}],\\
m_3 &= - \epsilon\, {}^3g^{\check r\check s} \Big( m_{2\check s} [\partial_{\check r} m_1 - \tilde \lambda_{2\tau \check r}] - m_{1\check s} [\partial_{\check r} m_2 - \frac{1}{2} \tilde \lambda_{2\tau \check r}] +\\
&+ \partial_{\check r} m_2 [\tilde \lambda_{1\check s} + \frac{1}{2} \tilde \lambda_{1\check s\check u} \sigma^{\check u}] - \partial_{\check r} m_1 [\tilde \lambda_{2\check s} + \frac{1}{2} \tilde \lambda_{2\check s\check u} \sigma^{\check u}] \Big) -\\
&- \frac{\epsilon}{2} ({}^3g^{\check r\check s} - \delta^{\check r\check s}) \Big( \tilde \lambda_{1\tau \check r} [\tilde \lambda_{2\check s} + \frac{1}{2} \tilde \lambda_{2\check s\check u} \sigma^{\check u}] - \tilde \lambda_{2\tau \check r} [\tilde \lambda_{1\check s} + \frac{1}{2} \tilde \lambda_{1\check s\check u} \sigma^{\check u}] \Big),\\
\tilde \lambda_{3\check r} &= \frac{1}{2} \Big( \tilde \lambda_{1\tau} \tilde \lambda_{2\tau \check r} - \tilde \lambda_{2\tau} \tilde \lambda_{1\tau \check r} - \epsilon \delta^{\check m\check n} [\tilde \lambda_{1\check r\check m} \tilde \lambda_{2\check n} - \tilde \lambda_{2\check r\check m} \tilde \lambda_{1\check n}] \Big),\\
\tilde \lambda_{3\check r\check u} &= \frac{1}{2} \Big( \tilde \lambda_{1\tau \check u} \tilde \lambda_{2\tau \check r} - \tilde \lambda_{2\tau \check u} \tilde \lambda_{1\tau \check r} - \epsilon \delta^{\check m\check n} [\tilde \lambda_{1\check r\check m} \tilde \lambda_{2\check n\check u} - \tilde \lambda_{2\check r\check m} \tilde \lambda_{1\check n\check u}] \Big),\\
m_{3\check r} &= m_2 [\partial_{\check r} m_1 - \frac{1}{2} \tilde \lambda_{1\tau \check r}] - m_1 [\partial_{\check r} m_2 - \frac{1}{2} \tilde \lambda_{2\tau \check r}] +\\
&+ \partial_{\check r} m_2 [\tilde \lambda_{1\tau} + \frac{1}{2} \tilde \lambda_{1\tau \check u} \sigma^{\check u}] - \partial_{\check r} m_1 [\tilde \lambda_{2\tau} + \frac{1}{2} \tilde \lambda_{2\tau \check u} \sigma^{\check u}] -\\
&- \frac{\epsilon}{2} ({}^3g^{\check m\check n} - \delta^{\check m\check n}) \Big( [\tilde \lambda_{1\check m} + \frac{1}{2} \tilde \lambda_{1\check m\check u} \sigma^{\check u}] \tilde \lambda_{2\check r\check n} - [\tilde \lambda_{2\check m} + \frac{1}{2} \tilde \lambda_{2\check m\check u} \sigma^{\check u}] \tilde \lambda_{1\check r\check n} \Big) -
\end{aligned}
$$

$$
\begin{aligned}
&-\epsilon^3 g^{\check m\check n}\Big(m_{1\check m}[\partial_{\check n} m_{2\check r} - \frac{1}{2}\tilde\lambda_{2\check r\check n}] - m_{2\check m}[\partial_{\check n} m_{1\check r} - \frac{1}{2}\tilde\lambda_{1\check r\check n}] +\\
&+\frac{1}{2}[\partial_{\check m} m_{1\check r}\tilde\lambda_{2\check n\check u} - \partial_{\check m} m_{2\check r}\tilde\lambda_{1\check n\check u}]\sigma^{\check u}\Big) -\\
&-\epsilon^3 g_{\check r\check s}\,{}^3g^{\check t\check n}\,\partial_{\check t}\,{}^3g^{\check s\check m}\Big([m_{1\check n} - \tilde\lambda_{1\check n} - \frac{1}{2}\tilde\lambda_{1\check n\check u}\sigma^{\check u}][m_{2\check m} - \tilde\lambda_{2\check m} - \frac{1}{2}\tilde\lambda_{2\check m\check u}\sigma^{\check u}] -\\
&-[m_{2\check n} - \tilde\lambda_{2\check n} - \frac{1}{2}\tilde\lambda_{2\check n\check u}\sigma^{\check u}][m_{1\check m} - \tilde\lambda_{1\check m} - \frac{1}{2}\tilde\lambda_{1\check m\check u}\sigma^{\check u}]\Big),
\end{aligned}
$$

$$
\begin{aligned}
\int d^3\sigma\quad &[m_3\tilde{\mathcal H} + m_3^{\check r}\,{}^3\tilde{\mathcal H}_{\check r}](\tau,\vec\sigma) + \tilde\lambda_{3A}(\tau)\hat P^A_{ADM} + \frac{1}{2}\tilde\lambda_{3AB}(\tau)\hat J^{AB}_{ADM} =\\
&= \int d^3\sigma_1 d^3\sigma_2\Big[m_1(\tau,\vec\sigma_1)m_2(\tau,\vec\sigma_2)\{\tilde{\mathcal H}(\tau,\vec\sigma_1),\tilde{\mathcal H}(\tau,\vec\sigma_2)\} +\\
&+ [m_1(\tau,\vec\sigma_1)m_2^{\check r}(\tau,\vec\sigma_2) - m_2(\tau,\vec\sigma_1)m_1^{\check r}(\tau,\vec\sigma_2)]\{\tilde{\mathcal H}(\tau,\vec\sigma_1),{}^3\tilde{\mathcal H}_{\check r}(\tau,\vec\sigma_2)\} +\\
&+ m_1^{\check r}(\tau,\vec\sigma_1)m_2^{\check s}(\tau,\vec\sigma_2)\{{}^3\tilde{\mathcal H}_{\check r}(\tau,\vec\sigma_1),{}^3\tilde{\mathcal H}_{\check s}(\tau,\vec\sigma_2)\}\Big] +\\
&+ \int d^3\sigma\Big[\Big(\tilde\lambda_{1A}(\tau)m_2(\tau,\vec\sigma) - \tilde\lambda_{2A}(\tau)m_1(\tau,\vec\sigma)\Big)\{\hat P^A_{ADM},\tilde{\mathcal H}(\tau,\vec\sigma)\} +\\
&+ \Big(\tilde\lambda_{1A}(\tau)m_2^{\check r}(\tau,\vec\sigma) - \tilde\lambda_{2A}(\tau)m_1^{\check r}(\tau,\vec\sigma)\Big)\{\hat P^A_{ADM},{}^3\tilde{\mathcal H}_{\check r}(\tau,\vec\sigma)\} +\\
&+ \frac{1}{2}\Big(\tilde\lambda_{1AB}(\tau)m_2(\tau,\vec\sigma) - \tilde\lambda_{2AB}(\tau)m_1(\tau,\vec\sigma)\Big)\{\hat J^{AB}_{ADM},\tilde{\mathcal H}(\tau,\vec\sigma)\} +\\
&+ \frac{1}{2}\Big(\tilde\lambda_{1AB}(\tau)m_2^{\check r}(\tau,\vec\sigma) - \tilde\lambda_{2AB}(\tau)m_1^{\check r}(\tau,\vec\sigma)\Big)\{\hat J^{AB}_{ADM},{}^3\tilde{\mathcal H}_{\check r}(\tau,\vec\sigma)\}\Big] +\\
&+ \tilde\lambda_{1A}(\tau)\tilde\lambda_{2B}(\tau)\{\hat P^A_{ADM},\hat P^B_{ADM}\} + \frac{1}{4}\tilde\lambda_{1AB}(\tau)\tilde\lambda_{2CD}(\tau)\{\hat J^{AB}_{ADM},\hat J^{CD}_{ADM}\} +\\
&+ \frac{1}{2}\Big(\tilde\lambda_{1A}(\tau)\tilde\lambda_{2CD}(\tau) - \tilde\lambda_{2A}(\tau)\tilde\lambda_{1CD}(\tau)\Big)\{\hat P^A_{ADM},\hat J^{CD}_{ADM}\}. \qquad (15)
\end{aligned}
$$

This implies:
i) the Poisson brackets of two proper gauge transformations [$\tilde\lambda_{iA} = \tilde\lambda_{iAB} = 0$, i=1,2] is a proper gauge transformation [$\tilde\lambda_{3A} = \tilde\lambda_{3AB} = 0$], see Eq.(5);
ii) if $N_2 = m_2$, $N_{2\check r} = m_{2\check r}$ [$\tilde\lambda_{2A} = \tilde\lambda_{2AB} = 0$] correspond to a proper gauge transformation and $N_1, N_{1\check r}$ [$m_1 = m_{1\check r} = 0$] to an improper one, then we get a proper gauge transformation

$$
\begin{aligned}
&\tilde\lambda_{3A} = \tilde\lambda_{3AB} = 0,\\
&m_3 = -\epsilon^3 g^{\check r\check s}(-\tfrac{1}{2}m_{2\check s}\tilde\lambda_{1\tau\check r} + \partial_{\check r} m_2[\tilde\lambda_{1\check s} + \tfrac{1}{2}\tilde\lambda_{1\check s\check u}\sigma^{\check u}]),\\
&m_{3\check r} = -\tfrac{1}{2}m_2\tilde\lambda_{1\tau\check r} + \partial_{\check r} m_2[\tilde\lambda_{1\tau} + \tfrac{1}{2}\tilde\lambda_{1\check s\check u}\sigma^{\check u}] - \tfrac{\epsilon}{2}{}^3g^{\check m\check n}(m_{2\check m}\tilde\lambda_{1\check r\check n} - \partial_{\check m} m_{2\check r}\tilde\lambda_{1\check n\check u}\sigma^{\check u}) -\\
&-\epsilon^3 g_{\check r\check s}\,{}^3g^{\check t\check n}\partial_{\check t}\,{}^3g^{\check s\check m}(m_{2\check n}[\tilde\lambda_{1\check m} + \tfrac{1}{2}\tilde\lambda_{1\check m\check u}\sigma^{\check u}] - m_{2\check m}[\tilde\lambda_{1\check n} + \tfrac{1}{2}\tilde\lambda_{1\check n\check u}\sigma^{\check u}]),
\end{aligned}
$$

and Eqs.(15) may be interpreted as saying that the 10 Poincaré charges are 'gauge invariant' and Noether constants of motion

$$
\begin{aligned}
&\{\hat P^\tau_{ADM},\tilde{\mathcal H}(\tau,\vec\sigma)\} = -\partial_{\check r}\,{}^3\tilde{\mathcal H}^{\check r}(\tau,\vec\sigma) \approx 0,\\
&\{\hat P^\tau_{ADM},{}^3\tilde{\mathcal H}_{\check r}(\tau,\vec\sigma)\} = 0,\\
&\{\hat P^{\check r}_{ADM},\tilde{\mathcal H}(\tau,\vec\sigma)\} = \epsilon\partial_{\check s}[{}^3g^{\check r\check s}\,\tilde{\mathcal H}(\tau,\vec\sigma)] \approx 0,\\
&\{\hat P^{\check r}_{ADM},{}^3\tilde{\mathcal H}_{\check s}(\tau,\vec\sigma)\} = -\epsilon\partial_{\check s}\,{}^3g^{\check r\check t}(\tau,\vec\sigma)\,{}^3\tilde{\mathcal H}_{\check t}(\tau,\vec\sigma) +\\
&+\epsilon^3 g^{\check r\check t}(\tau,\vec\sigma)\,{}^3g_{\check s\check w}(\tau,\vec\sigma)\partial_{\check t}\,{}^3g^{\check w\check u}(\tau,\vec\sigma)\,{}^3\tilde{\mathcal H}_{\check u}(\tau,\vec\sigma) \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
&\{\hat J^{\tau\check r}_{ADM}, \tilde{\mathcal{H}}(\tau,\vec\sigma)\} = 2\, {}^3\tilde{\mathcal{H}}^{\check r}(\tau,\vec\sigma) - 2\partial_{\check s}[\sigma^{\check r}\, {}^3\tilde{\mathcal{H}}^{\check s}(\tau,\vec\sigma)] \approx 0,\\
&\{\hat J^{\tau\check r}_{ADM}, {}^3\tilde{\mathcal{H}}_{\check s}(\tau,\vec\sigma)\} = -\delta^{\check r}_{\check s}\, \tilde{\mathcal{H}}(\tau,\vec\sigma) \approx 0,\\
&\{\hat J^{\check r\check s}_{ADM}, \tilde{\mathcal{H}}(\tau,\vec\sigma)\} = \epsilon\partial_{\check u}\Big([{}^3g^{\check r\check u}\sigma^{\check s} - {}^3g^{\check s\check u}\sigma^{\check r}]\tilde{\mathcal{H}}\Big)(\tau,\vec\sigma) \approx 0,\\
&\{\hat J^{\check r\check s}_{ADM}, {}^3\tilde{\mathcal{H}}_{\check w}(\tau,\vec\sigma)\} = \Big((\delta^{\check r}_{\check u}\delta^{\check s}_{\check w} - \delta^{\check r}_{\check w}\delta^{\check s}_{\check u})^3\tilde{\mathcal{H}}^{\check u}(\tau,\vec\sigma) +\\
&+\sigma^{\check s}\Big[-\epsilon\partial_{\check w}\, {}^3g^{\check r\check t}\, {}^3\tilde{\mathcal{H}}_{\check t} + {}^3g_{\check w\check v}\, {}^3g^{\check r\check m}\partial_{\check m}\, {}^3\tilde{\mathcal{H}}^{\check v}\Big](\tau,\vec\sigma) -\\
&-\sigma^{\check r}\Big[-\epsilon\partial_{\check w}\, {}^3g^{\check s\check t}\, {}^3\tilde{\mathcal{H}}_{\check t} + {}^3g_{\check w\check v}\, {}^3g^{\check s\check m}\partial_{\check m}\, {}^3\tilde{\mathcal{H}}^{\check v}\Big](\tau,\vec\sigma)\Big) \approx 0,\\
&\Downarrow\\
&\partial_\tau\, \hat P^A_{ADM} \overset{\circ}{=} \{\hat P^A_{ADM}, \hat H^{'}_{(D)ADM}\} = \{\hat P^A_{ADM}, \hat H^{'}_{(c)ADM}\} \approx 0,\\
&\partial_\tau\, \hat J^{AB}_{ADM} \overset{\circ}{=} \{\hat J^{AB}_{ADM}, \hat H^{'}_{(D)ADM}\} = \{\hat J^{AB}_{ADM}, \hat H^{'}_{(c)ADM}\} \approx 0.
\end{aligned}
\tag{16}
$$

From Eqs.(16) we see that also the strong Poincaré charges are constants of motion [we did not succeeded to show that they are conserved independently from the first class constraints]

$$
\begin{aligned}
P^\tau_{ADM} &= \hat P^\tau_{ADM} + \int d^3\sigma \tilde{\mathcal{H}}(\tau,\vec\sigma),\\
P^{\check r}_{ADM} &= \hat P^{\check r}_{ADM} + \int d^3\sigma\, {}^3\tilde{\mathcal{H}}^{\check r}(\tau,\vec\sigma),\\
J^{\tau\check r}_{ADM} &= \hat J^{\tau\check r}_{ADM} + \frac{1}{2}\int d^3\sigma\sigma^{\check r}\, \tilde{\mathcal{H}}(\tau,\vec\sigma),\\
J^{\check r\check s}_{ADM} &= \hat J^{\check r\check s}_{ADM} + \int d^3\sigma[\sigma^{\check s}\, {}^3\tilde{\mathcal{H}}^{\check r}(\tau,\vec\sigma) - \sigma^{\check r}\, {}^3\tilde{\mathcal{H}}^{\check s}(\tau,\vec\sigma)],\\
&\Downarrow\\
&\partial_\tau\, P^A_{ADM} \approx 0,\\
&\partial_\tau\, J^{AB}_{ADM} \approx 0;
\end{aligned}
\tag{17}
$$

iii) the Poisson bracket of two improper gauge transformations [$m_i = m_{i\check r} = 0$, i=1,2] is an improper gauge transformation with the previous $\tilde\lambda_{3A}$, $\tilde\lambda_{3AB}$ and with

$$
\begin{aligned}
m_3 &= -\tfrac{\epsilon}{2}({}^3g^{\check r\check s} - \delta^{\check r\check s})(\tilde\lambda_{1\tau\check r}[\tilde\lambda_{2\check s} + \tfrac{1}{2}\tilde\lambda_{2\check s\check u}\sigma^{\check u}] - \tilde\lambda_{2\tau\check r}[\tilde\lambda_{1\check s} + \tfrac{1}{2}\tilde\lambda_{1\check s\check u}\sigma^{\check u}]),\\
m_{3\check r} &= -\tfrac{\epsilon}{2}({}^3g^{\check m\check n} - \delta^{\check m\check n})([\tilde\lambda_{1\check m} + \tfrac{1}{2}\tilde\lambda_{1\check m\check u}\sigma^{\check u}]\tilde\lambda_{2\check r\check n} - [\tilde\lambda_{2\check m} + \tfrac{1}{2}\tilde\lambda_{2\check m\check u}\sigma^{\check u}]\tilde\lambda_{1\check r\check n}).
\end{aligned}
$$

This implies that the 10 strong Poincaré charges [and, therefore, also the weak ones] satisfy the Poincaré algebra modulo the first class constraints, namely modulo the Hamiltonian group $\bar{\mathcal{G}}$ of gauge transformations

$$
\begin{aligned}
&\{\hat P^\tau_{ADM}, \hat J^{\tau\check r}_{ADM}\} = -\epsilon \hat P^{\check r}_{ADM},\\
&\{\hat P^\tau_{ADM}, \hat J^{\check r\check s}_{ADM}\} = 0,\\
&\{\hat P^{\check u}_{ADM}, \hat J^{\tau\check r}_{ADM}\} = -\epsilon\delta^{\check u\check r}\hat P^\tau_{ADM} + \epsilon\int d^3\sigma[({}^3g^{\check u\check r} - \delta^{\check u\check r})\tilde{\mathcal{H}}](\tau,\vec\sigma),\\
&\{\hat P^{\check u}_{ADM}, \hat J^{\check r\check s}_{ADM}\} = -\epsilon\Big[\delta^{\check u\check s}\hat P^{\check r}_{ADM} - \delta^{\check u\check r}\hat P^{\check s}_{ADM} +\\
&\qquad + \int d^3\sigma[({}^3g^{\check u\check s} - \delta^{\check u\check s})^3\tilde{\mathcal{H}} - ({}^3g^{\check u\check r} - \delta^{\check u\check r})^3\tilde{\mathcal{H}}^{\check s}](\tau,\vec\sigma)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\{\hat{J}^{\tau\check{r}}_{ADM}, \hat{J}^{\tau\check{s}}_{ADM}\} = \epsilon \hat{J}^{\check{r}\check{s}}_{ADM},\\
&\{\hat{J}^{\tau\check{r}}_{ADM}, \hat{J}^{\check{u}\check{v}}_{ADM}\} = \epsilon\Big[\delta^{\check{r}\check{u}}\hat{J}^{\tau\check{v}}_{ADM} - \delta^{\check{r}\check{v}}\hat{J}^{\tau\check{u}}_{ADM} -\\
&\qquad - \int d^3\sigma[\Big(\sigma^{\check{v}}({}^3g^{\check{r}\check{u}} - \delta^{\check{r}\check{u}}) - \sigma^{\check{u}}({}^3g^{\check{r}\check{v}} - \delta^{\check{r}\check{v}})\Big)\tilde{\mathcal{H}}](\tau,\vec{\sigma})\Big],\\
&\{\hat{J}^{\check{r}\check{s}}_{ADM}, \hat{J}^{\check{u}\check{v}}_{ADM}\} = -\epsilon[\delta^{\check{r}\check{u}}\hat{J}^{\check{s}\check{v}}_{ADM} + \delta^{\check{s}\check{v}}\hat{J}^{\check{r}\check{u}}_{ADM} - \delta^{\check{r}\check{v}}\hat{J}^{\check{s}\check{u}}_{ADM} - \delta^{\check{s}\check{u}}\hat{J}^{\check{r}\check{v}}_{ADM}] +\\
&\qquad + \epsilon\int d^3\sigma\Big[\Big(\sigma^{\check{s}}({}^3g^{\check{r}\check{v}} - \delta^{\check{r}\check{v}}) - \sigma^{\check{r}}({}^3g^{\check{s}\check{v}} - \delta^{\check{s}\check{v}})\Big){}^3\tilde{\mathcal{H}}^{\check{u}} +\\
&\qquad + \Big(\sigma^{\check{u}}({}^3g^{\check{v}\check{s}} - \delta^{\check{v}\check{s}}) - \sigma^{\check{v}}({}^3g^{\check{u}\check{s}} - \delta^{\check{u}\check{s}})\Big){}^3\tilde{\mathcal{H}}^{\check{r}} -\\
&\qquad - \Big(\sigma^{\check{s}}({}^3g^{\check{r}\check{u}} - \delta^{\check{r}\check{u}}) - \sigma^{\check{r}}({}^3g^{\check{s}\check{u}} - \delta^{\check{s}\check{u}})\Big){}^3\tilde{\mathcal{H}}^{\check{v}} -\\
&\qquad - \Big(\sigma^{\check{u}}({}^3g^{\check{r}\check{v}} - \delta^{\check{r}\check{v}}) - \sigma^{\check{v}}({}^3g^{\check{u}\check{r}} - \delta^{\check{u}\check{r}})\Big){}^3\tilde{\mathcal{H}}^{\check{s}}\Big](\tau,\vec{\sigma}),\\
&\qquad \Downarrow\\
&\{\hat{P}^A_{ADM}, \hat{P}^B_{ADM}\} = 0,\\
&\{\hat{P}^A_{ADM}, \hat{J}^{BC}_{ADM}\} \approx {}^4\eta^{AC}\hat{P}^B_{ADM} - {}^4\eta^{AB}\hat{P}^C_{ADM},\\
&\{\hat{J}^{AB}_{ADM}, \hat{J}^{CD}_{ADM}\} \approx -C^{ABCD}_{EF}\hat{J}^{EF}_{ADM},
\end{aligned}
\tag{18}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\{P^A_{ADM}, P^B_{ADM}\} \approx 0,\\
&\{P^A_{ADM}, J^{BC}_{ADM}\} \approx {}^4\eta^{AC}P^B_{ADM} - {}^4\eta^{AB}P^C_{ADM},\\
&\{J^{AB}_{ADM}, J^{CD}_{ADM}\} \approx -C^{ABCD}_{EF}J^{EF}_{ADM},
\end{aligned}
\tag{19}
$$

in accord with Eqs. (9).

In Ref. [6] it is noted that the terms depending on the constraints in Eq.(18) contain the Hamiltonian version of the supertranslation ambiguity. Indeed, these terms depend on ${}^3g^{\check{r}\check{s}}(\tau,\vec{\sigma}) - \delta^{\check{r}\check{s}}$ and, by using Eq.(2), this quantity may be rewritten as $-\frac{1}{r}\,{}^3\tilde{s}^{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) + {}^3\tilde{g}^{\check{r}\check{s}}(\tau,\vec{\sigma})$ with ${}^3\tilde{g}^{\check{r}\check{s}}(\tau,\vec{\sigma})$ going to zero at spatial infinity faster than $1/r$. Now the objects $\int d^3\sigma\, \frac{1}{r}\,{}^3\tilde{s}^{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r})\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$,.... are generators of supertranslations with zero momentum generalizing those [i.e. $\int d^3\sigma[s\tilde{\mathcal{H}} + s_{\check{r}}\,{}^3\tilde{\mathcal{H}}](\tau,\vec{\sigma})$] appearing in the Dirac Hamiltonian.

To remove this gauge ambiguity in the Poincaré algebra and simultaneously to kill the supertranslations, which forbid the existence of a unique Poincaré group, the strategy of Ref. [6] is to add four gauge-fixings to the secondary first class constraints $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{\mathcal{H}}^{\check{r}}(\tau,\vec{\sigma}) \approx 0$ to fix a coordinate system and therefore to build a realization of the reduced phase space. In Appendix B there is a review of the main realizations existing in the literature for metric gravity (some of them have been already quoted in Section II). In Ref. [6] one uses the maximal slice condition and harmonic 3-coordinates.

Anderson's paper [30], quoted in Section II, shows that to have "zero momentum" for the supertranslations [namely vanishing supertranslation charges arising from the parts $s(\tau,\vec{\sigma})$, $s_{\check{r}}(\tau,\vec{\sigma})$ of $n(\tau,\vec{\sigma})$, $n_{\check{r}}(\tau,\vec{\sigma})$] and also to have well defined Lorentz charges, one needs the parity conditions in suitable function spaces, which do not imply a strong Poincaré algebra, and a class $\mathcal{C}$ of coordinate systems of $M^4$ including the gauges corresponding to York QI gauge conditions. In that paper it is also shown that to preserve the boundary conditions containing the parity conditions, one has to restrict $Diff\, M^4$ to the allowed transformations

$Diff_I\, M^4 \times P$ [namely to pseudo-diffeomorphisms tending to the identity in a direction-independent way at spatial infinity plus the Poincaré group].

Instead of adding gauge fixings, we shall assume the existence of a restricted class $\mathcal{C}$ of coordinate systems for $M^4$ associated with Eqs.(10) [i.e. corresponding to $s(\tau,\vec{\sigma}) = s_{\check{r}}(\tau,\vec{\sigma}) = {}^3t^{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) = 0$, ${}^3s_{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) = M\delta_{\check{r}\check{s}}$] and that the gauge transformations are so restricted that we cannot leave this class $\mathcal{C}$. The four gauge-fixings then allow to choose a particular coordinate system in the class $\mathcal{C}$ and to get a strong Poincaré algebra.

Since supertranslations must be absent to have a unique Poincaré algebra, it must be

$$s(\tau,\vec{\sigma}) = s_{\check{r}}(\tau,\vec{\sigma}) = 0, \text{ namely } n(\tau,\vec{\sigma}) = m(\tau,\vec{\sigma}),\ n_{\check{r}}(\tau,\vec{\sigma}) = m_{\check{r}}(\tau,\vec{\sigma}),$$

in every allowed coordinate system. This suggests that, in a suitable class $\mathcal{C}$ of coordinate systems for $M^4$ [then transformed to coordinates adapted to the 3+1 splitting of $M^4$ with a foliation with spacelike leaves $\Sigma_\tau$, whose allowed coordinates systems are in the previously defined atlas $\mathcal{C}_\tau$] asymptotic to Minkowski coordinates and with the general coordinate transformations suitably restricted at spatial infinity so that it is not possible to go out this class, one should have the following direction-independent boundary conditions for the ADM variables for $r \rightarrow \infty$ [$\epsilon > 0$]

$$\begin{aligned}
&{}^3g_{\check{r}\check{s}}(\tau,\vec{\sigma}) = (1 + \frac{M}{r})\delta_{\check{r}\check{s}} + {}^3h_{\check{r}\check{s}}(\tau,\vec{\sigma}), \qquad {}^3h_{\check{r}\check{s}}(\tau,\vec{\sigma}) = O(r^{-(1+\epsilon)}),\\
&{}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau,\vec{\sigma}) = {}^3k^{\check{r}\check{s}}(\tau,\vec{\sigma}) = O(r^{-(2+\epsilon)}),
\end{aligned}$$

$$\begin{aligned}
N(\tau,\vec{\sigma}) &= N_{(as)}(\tau,\vec{\sigma}) + n(\tau,\vec{\sigma}), \qquad n(\tau,\vec{\sigma}) = O(r^{-(2+\epsilon)}),\\
N_{\check{r}}(\tau,\vec{\sigma}) &= N_{(as)\check{r}}(\tau,\vec{\sigma}) + n_{\check{r}}(\tau,\vec{\sigma}), \qquad n_{\check{r}}(\tau,\vec{\sigma}) = O(r^{-\epsilon}),
\end{aligned}$$

$$\begin{aligned}
N_{(as)}(\tau,\vec{\sigma}) &= -\tilde{\lambda}_\tau(\tau) - \frac{1}{2}\tilde{\lambda}_{\tau\check{s}}(\tau)\sigma^{\check{s}},\\
N_{(as)\check{r}}(\tau,\vec{\sigma}) &= -\tilde{\lambda}_{\check{r}}(\tau) - \frac{1}{2}\tilde{\lambda}_{\check{r}\check{s}}(\tau)\sigma^{\check{s}},\\
\Rightarrow \qquad & N_{(as)A}(\tau,\vec{\sigma}) \stackrel{def}{=} (N_{(as)}\,;\, N_{(as)\check{r}})(\tau,\vec{\sigma}) = -\tilde{\lambda}_A(\tau) - \frac{1}{2}\tilde{\lambda}_{A\check{s}}(\tau)\sigma^{\check{s}},
\end{aligned} \tag{20}$$

in accord with Regge-Teitelboim [5] and Beig-O'Murchadha [6]. We have assumed the angle-independent behaviour ${}^3s_{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) = M\delta_{\check{r}\check{s}}$, ${}^3t^{\check{r}\check{s}}(\tau, \frac{\sigma^{\check{n}}}{r}) = 0$. Since this implies the vanishing of the ADM momentum, $P^{\check{r}}_{ADM} = 0$, we see that the elimination of supertranslations seems to be connected with a definition of "rest frame" in the asymptotic Dirac coordinates $z^{(\mu)}_{(\infty)}(\tau,\vec{\sigma})$. Therefore, the previous boundary conditions on ${}^3g$, ${}^3\tilde{\Pi}$, are compatible and can be replaced with the Christodoulou-Klainermann ones of Eq.(1). To have a non-vanishing ADM momentum one should have ${}^3t^{\check{r}\check{s}}(\tau,\vec{\sigma}) = const.\, \delta^{\check{r}\check{s}}$ in Eqs.(3) violating the parity conditions and creating problems with supertranslations.

However, now Eq.(17) and $P^{\check{r}}_{ADM} = 0$ imply

$$\begin{aligned}
&\hat{P}^{\check{r}}_{ADM} \approx 0,\\
&P^{(\mu)}_{ADM} = b^{(\mu)}_{(\infty)\tau} P^\tau_{ADM} = l^{(\mu)}_{(\infty)} P^\tau_{ADM},\\
&\hat{P}^{(\mu)}_{ADM} \approx l^{(\mu)}_{(\infty)} \hat{P}^\tau_{ADM}.
\end{aligned} \tag{21}$$

But, as we have said, these results may be obtained in parametrized Minkowski theories only after having done the restriction to Wigner-like hypersurfaces by adding 6 suitable gauge fixing constraints, whose time constancy implies $\tilde{\lambda}_{(\mu)(\nu)}(\tau) = 0$ so that $\tilde{\lambda}_{AB}(\tau) = 0$.

With these assumptions one has from Eqs.(6) of I the following form of the line element

$$
\begin{aligned}
ds^2 &= \epsilon\Big([N_{(as)} + n]^2 - [N_{(as)\check{r}} + n_{\check{r}}]^3 g^{\check{r}\check{s}} [N_{(as)\check{s}} + n_{\check{s}}]\Big)(d\tau)^2 - \\
&- 2\epsilon [N_{(as)\check{r}} + n_{\check{r}}] d\tau d\sigma^{\check{r}} - \epsilon\, {}^3g_{\check{r}\check{s}} d\sigma^{\check{r}} d\sigma^{\check{s}} = \\
&= \epsilon\Big([N_{(as)} + n]^2 (d\tau)^2 - \\
&\quad {}^3g^{\check{r}\check{s}} [{}^3g_{\check{r}\check{u}} d\sigma^{\check{u}} + (N_{(as)\check{r}} + n_{\check{r}}) d\tau][{}^3g_{\check{s}\check{v}} d\sigma^{\check{v}} + (N_{(as)\check{s}} + n_{\check{s}}) d\tau]\Big).
\end{aligned}
\tag{22}
$$

The Dirac Hamiltonian without supertranslations is

$$
\begin{aligned}
\hat{H}^{"}_{(c)ADM} &= \int d^3\sigma [(N_{(as)} + n)\tilde{\mathcal{H}} + (N_{(as)\check{r}} + n_{\check{r}})\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau, \vec{\sigma}) + \\
&+ \tilde{\lambda}_A(\tau) P^A_{ADM} + \frac{1}{2} \tilde{\lambda}_{AB}(\tau) J^{AB}_{ADM} = \\
&= \int d^3\sigma [n\tilde{\mathcal{H}} + n_{\check{r}}\, {}^3\tilde{\mathcal{H}}^{\check{r}}](\tau, \vec{\sigma}) + \tilde{\lambda}_A(\tau) \hat{P}^A_{ADM} + \frac{1}{2} \tilde{\lambda}_{AB}(\tau) \hat{J}^{AB}_{ADM}, \\
\hat{H}^{"}_{(D)ADM} &= \hat{H}^{"}_{(c)ADM} + \int d^3\sigma [\lambda_n \tilde{\pi}^n + \lambda^{\vec{n}}_r \tilde{\pi}^r_{\vec{n}}](\tau, \vec{\sigma}) + \zeta_A(\tau) \tilde{\pi}^A(\tau) + \zeta_{AB}(\tau) \tilde{\pi}^{AB}(\tau),
\end{aligned}
\tag{23}
$$

The conclusion of this discussion is a qualitative indication on the choice of of the special class $\mathcal{C}$ of coordinate systems on $M^4$ and of the function space $\mathcal{W}$ [an appropriate weighted Sobolev space as for Yang-Mills theory [40]] for the field variables ${}^3g_{\check{r}\check{s}}(\tau, \vec{\sigma})$, ${}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau, \vec{\sigma})$, $n(\tau, \vec{\sigma})$, $n_{\check{r}}(\tau, \vec{\sigma})$ and for the parameters $\alpha(\tau, \vec{\sigma})$, $\alpha_{\check{r}}(\tau, \vec{\sigma})$ [of which $n(\tau, \vec{\sigma})$, $n_{\check{r}}(\tau, \vec{\sigma})$ are special cases] of allowed proper gauge transformations connected to the identity [the rigid improper ones have been eliminated and replaced by the new canonical variables $\tilde{\lambda}_A(\tau)$, $\tilde{\lambda}_{AB}(\tau)$], generated by the secondary first class constraints.

We must have:

i) The atlas $\mathcal{C}$ for $M^4$ equipped with the 3+1 splittings should contain only coordinate systems approaching the Dirac asymptotic Minkowski rectangular coordinates of Eq.(8) at spatial infinity in a direction-independent way.

ii) The allowed 3+1 splittings must have the leaves, i.e. the Cauchy spacelike hypersurfaces $\Sigma_\tau$, approaching Minkowski hyperplanes at spatial infinity in a direction-independent way. The leaves $\Sigma_\tau \approx R^3$ have an atlas $\mathcal{C}_\tau$ containing the global coordinate systems $\{\sigma^{\check{r}}\}$ in which Eq.(20) holds.

iii)As a consequence of what has been said and of Eqs.(20), the space $\mathcal{W}$ should be defined by angle (or direction)-independent boundary conditions for the field variables for $r \rightarrow \infty$ of the following form:

$$
\begin{aligned}
&{}^3g_{\check{r}\check{s}}(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} (1 + \frac{M}{r})\delta_{\check{r}\check{s}} + {}^3h_{\check{r}\check{s}}(\tau, \vec{\sigma}) = (1 + \frac{M}{r})\delta_{\check{r}\check{s}} + o_4(r^{-3/2}), \\
&{}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} {}^3k^{\check{r}\check{s}}(\tau, \vec{\sigma}) = o_3(r^{-5/2}), \\
&n(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-(2+\epsilon)}), \quad \epsilon > 0, \\
&n_{\check{r}}(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}), \quad \epsilon > 0, \\
&\tilde{\pi}_n(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-3}),
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\pi}^{\check{r}}_{\vec{n}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-3}),\\
&\lambda_n(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-(3+\epsilon)}),\\
&\lambda^{\vec{n}}_{\check{r}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
&\alpha(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-(3+\epsilon)}),\\
&\alpha_{\check{r}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
&\Downarrow\\
&\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-3}),\\
&{}^3\tilde{\mathcal{H}}^{\check{r}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-3}).
\end{aligned}
\tag{24}
$$

With these boundary conditions we have $\partial_{\check{u}}\, {}^3g_{\check{r}\check{s}} = O(r^{-2})$ and not $O(r^{-(1+\epsilon)})$ [note that with this last condition and $\epsilon < 1/2$ it is shown in Ref. [64] that the ADM action (but in the first order formulation) becomes meaningless since the spatial integral diverges (in this reference it is also noted that with these boundary conditions adapted to asymptotic flatness at spatial infinity the Hilbert action may not produce a consistent and finite variational principle)]; this is compatible with the definition of gravitational radiation given by Christodoulou and Klainermann, but not with the one of Ref. [65].

In this function space $\mathcal{W}$ supertranslations are not allowed by definition and proper gauge transformations generated by the secondary constraints map $\mathcal{W}$ into itself. A coordinate-independent characterization of $\mathcal{W}$ should be given through an intrinsic definition of a minimal atlas of coordinate charts $\mathcal{C}_\tau$ of $\Sigma_\tau$ such that the lifts to 3-tensors on $\Sigma_\tau$ in $\mathcal{W}$ of the 3-diffeomorphisms in $Diff\, \Sigma_\tau$ maps them into them. Therefore, a unique asymptotic Poincaré group, modulo gauge transformations, is selected. Moreover, in accord with Anderson [30] also $Diff\, M^4$ is restricted to $Diff_I\, M^4 \times P$, so to map the class $\mathcal{C}$ of coordinate systems into itself. Now in $Diff_I\, M^4 \times P$ the allowed proper pseudo-diffeomorphisms $Diff_I\, M^4$ are a normal subgroup (they go to the identity in an angle-independent way at spatial infinity), while the Poincaré group $P$ describes the rigid improper gauge transformations (the non-rigid improper ones are assumed to be absent) as in the quoted Bergmann proposal. Finally, following Marolf, the Poincaré group is not interpreted as a group of improper gauge transformations but only as a source of superselection rules, which however seem to be consistent only in the rest frame $P^{\check{r}}_{ADM} = 0$, if we insist on the absence of supertranslations so to have the possibility to define the ADM spin Casimir.

To summarize this discussion, after the modification of metric gravity at the canonical level with the addition of the surface integrals and with the primary constraints resulting from the assumed splitting of the lapse and shift functions, two possible scenarios can be imagined (for the second one the Lagrangian is unknown):

a) Consider as configurational variables

$$
n_A(\tau,\vec{\sigma}) = (n\,;\, n_{\check{r}}\,)(\tau,\vec{\sigma}),\ \tilde{\lambda}_A(\tau),\ \tilde{\lambda}_{AB}(\tau),\ {}^3g_{\check{r}\check{s}}(\tau,\vec{\sigma}),
$$

with conjugate momenta

$$
\tilde{\pi}^A_n(\tau,\vec{\sigma}) = (\tilde{\pi}^n\,;\, \tilde{\pi}^{\check{r}}_{\vec{n}})(\tau,\vec{\sigma}) \approx 0,\ \tilde{\pi}^A(\tau) \approx 0,\ \tilde{\pi}^{AB}(\tau) \approx 0,\ {}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau,\vec{\sigma})
$$

[the vanishing momenta are assumed to be the primary constraints], and take the following

Dirac Hamiltonian [it is finite and differentiable] as the defining Hamiltonian:

$$\hat{H}^{(1)}_{(D)ADM} = \int d^3\sigma [n_A\, \tilde{\mathcal{H}}^A + \lambda_{n\,A} \tilde{\pi}^A_n](\tau,\vec{\sigma}) + \tilde{\lambda}_A(\tau)\hat{P}^A_{ADM} + \frac{1}{2}\tilde{\lambda}_{AB}(\tau)\hat{J}^{AB}_{ADM} + \\ + \zeta_A(\tau)\tilde{\pi}^A(\tau) + \zeta_{AB}(\tau)\tilde{\pi}^{AB}(\tau), \tag{25}$$

where $n_A = (n; n_{\check{r}})$, $\tilde{\mathcal{H}}^A = (\tilde{\mathcal{H}}\, ;\, {}^3\tilde{\mathcal{H}}^{\check{r}}\,)$ and where $\lambda_{n\,A}(\tau,\vec{\sigma}) = (\lambda_n\, ;\, \lambda^{\vec{n}}_{\check{r}}\,)(\tau,\vec{\sigma})$, $\zeta_A(\tau)$, $\zeta_{AB}(\tau)$, are Dirac multipliers associated with the primary constraints.

For $\tilde{\lambda}_{AB}(\tau) = 0$, $\tilde{\lambda}_A(\tau) = \epsilon\delta_{A\tau}$, one has $\hat{H}^{(1)}_{(D)ADM} \approx \epsilon\hat{P}^{\tau}_{ADM}$ [51].

The time constancy of the primary constraints implies the following secondary ones

$\tilde{\mathcal{H}}^A(\tau,\vec{\sigma}) \approx 0$ [generators of proper gauge transformations],
$\hat{P}^A_{ADM} \approx 0$, $\hat{J}^{AB}_{ADM} \approx 0$

[either generators of improper gauge transformations (in this case 10 conjugate degrees of freedom in the 3-metric are extra gauge variables) or, following Marolf's proposal [45], defining a superselection sector (like it happens for the vanishing of the color charges for the confinement of quarks)], all of which are constants of the motion. All the constraints are first class, so that:

i) $\tilde{\lambda}_A(\tau)$, $\tilde{\lambda}_{AB}(\tau)$ are arbitrary gauge variables conjugate to $\tilde{\pi}^A(\tau) \approx 0$, $\tilde{\pi}^{AB}(\tau) \approx 0$;
ii) the physical reduced phase space of canonical metric gravity is restricted to have "zero asymptotic Poincaré charges" so that there is no natural Hamiltonian for the evolution in $\tau$.

This is the natural interpretation of ADM metric gravity which leads to the Wheeler-DeWitt equation after quantization (see the Conclusions for the problem of time in this scenario) and, in a sense, it is a Machian formulation of an asymptotically flat noncompact (with boundary) spacetime $M^4$ in the same spirit of Barbour's approach [47] and of the closed (without boundary) Einstein-Wheeler universes. However, in this case there is no solution to the problem of deparametrization of metric gravity and no connection with parametrized Minkowski theories restricted to spacelike hyperplanes.

Let us remark that the scenario a) corresponds to the exceptional orbit $\hat{P}^A_{ADM} = 0$ of the asymptotic Poincaré group.

b) According to the suggestion of Dirac, modify ADM metric gravity by adding the 10 new canonical pairs $x^{(\mu)}_{(\infty)}(\tau)$, $p^{(\mu)}_{(\infty)}$, $b^{(\mu)}_{(\infty)A}(\tau)$, $S^{(\mu)(\nu)}_{\infty}$ [with the previously given Dirac brackets implying the orthonormality constraints for the b's] to the metric gravity phase space with canonical basis $n_A(\tau,\vec{\sigma}) = (n\, ;\, n_{\check{r}}\,)(\tau,\vec{\sigma})$, $\tilde{\pi}^A_n(\tau,\vec{\sigma}) = (\tilde{\pi}^n; \tilde{\pi}^{\check{r}}_{\vec{n}}) \approx 0$ (the primary constraints), ${}^3g_{\check{r}\check{s}}(\tau,\vec{\sigma})$, ${}^3\tilde{\Pi}^{\check{r}\check{s}}(\tau,\vec{\sigma})$, and then:
i) add the 10 new primary constraints

$$\chi^A = p^A_{(\infty)} - \hat{P}^A_{ADM} = b^A_{(\infty)(\mu)}(\tau)[p^{(\mu)}_{(\infty)} - b^{(\mu)}_{(\infty)B}(\tau)\hat{P}^B_{ADM}] \approx 0,$$

$$\chi^{AB} = J^{AB}_{(\infty)} - \hat{J}^{AB}_{ADM} = b^A_{(\infty)(\mu)}(\tau) b^B_{(\infty)(\nu)}(\tau)[S^{(\mu)(\nu)}_{(\infty)} - b^{(\mu)}_{(\infty)C}(\tau) b^{(\nu)}_{(\infty)D}(\tau)\hat{J}^{CD}_{ADM}] \approx 0,$$

$$\{\chi^A, \chi^{BC}\} \approx {}^4\eta^{AC}\chi^B - {}^4\eta^{AB}\chi^C \approx 0 \qquad , \{\chi^A, \chi^B\} \approx 0,$$
$$\{\chi^{AB}, \chi^{CD}\} \approx -C^{ABCD}_{EF}\chi^{EF} \approx 0.$$

$$\{\chi^A(\tau), \tilde{\pi}^D_n(\tau, \vec{\sigma})\} = \{\chi^{AB}(\tau), \tilde{\pi}^D_n(\tau, \vec{\sigma})\} = 0,$$
$$\{\chi^A(\tau), \tilde{\mathcal{H}}^D(\tau, \vec{\sigma})\} \approx 0, \qquad \{\chi^{AB}(\tau), \tilde{\mathcal{H}}^D(\tau, \vec{\sigma})\} \approx 0,$$

where $p^A_{(\infty)} = b^A_{(\infty)(\mu)} p^{(\mu)}_{(\infty)}$, $J^{AB}_{(\infty)} = b^A_{(\infty)(\mu)} b^B_{(\infty)(\nu)} S^{(\mu)(\nu)}_{(\infty)}$ [remember that $p^A_{(\infty)}$ and $J^{AB}_{(\infty)}$ satisfy a Poincaré algebra];
ii) consider $\tilde{\lambda}_A(\tau)$, $\tilde{\lambda}_{AB}(\tau)$, as Dirac multipliers [like $\lambda_{nA}(\tau, \vec{\sigma})$] for these 10 new primary constraints, and not as configurational (arbitrary gauge) variables coming from the lapse and shift functions [so that there are no conjugate momenta $\tilde{\pi}^A(\tau)$, $\tilde{\pi}^{AB}(\tau)$ and no associated Dirac multipliers $\zeta_A(\tau)$, $\zeta_{AB}(\tau)$], in the assumed Dirac Hamiltonian [it is finite and differentiable]

$$H_{(D)ADM} = \int d^3\sigma [n_A \tilde{\mathcal{H}}^A + \lambda_{nA}\tilde{\pi}^A_n](\tau, \vec{\sigma}) -$$
$$- \tilde{\lambda}_A(\tau)[p^A_{(\infty)} - \hat{P}^A_{ADM}] - \frac{1}{2}\tilde{\lambda}_{AB}(\tau)[J^{AB}_{(\infty)} - \hat{J}^{AB}_{ADM}] \approx 0, \tag{26}$$

Now the reduced phase space is the ADM one and there is consistency with Marolf's proposal regarding superselection sectors: on the ADM variables there are only the secondary first class constraints $\tilde{\mathcal{H}}^A(\tau, \vec{\sigma}) \approx 0$ [generators of proper gauge transformations], because the other first class constraints $p^A_{(\infty)} - \hat{P}^A_{ADM} \approx 0$, $J^{AB}_{(\infty)} - \hat{J}^{AB}_{ADM} \approx 0$ do not generate improper gauge transformations but eliminate 10 of the extra 20 variables. One has an asymptotically flat at spatial infinity noncompact (with boundary $S_\infty$) spacetime $M^4$ with non-vanishing asymptotic Poincaré charges and the possibility to deparametrize metric gravity so to obtain the connection with parametrized Minkowski theories restricted to spacelike hyperplanes [more exactly to Wigner hyperplanes due to the rest-frame condition $P^{\check{r}}_{ADM} = 0$ forced by the elimination of supertranslations].

While the gauge-fixings for the primary constraints $\tilde{\Pi}^A_n(\tau, \vec{\sigma}) \approx 0$ and the resulting ones for the secondary ones $\tilde{\mathcal{H}}^A(\tau, \vec{\sigma}) \approx 0$, implying the determination of the $\lambda_{nA}(\tau, \vec{\sigma})$, follow the scheme outlined at the end of Section V of I and in the Conclusions of II, one has to clarify the meaning of the gauge-fixings for the extra 10 first class constraints.

Let us remark that the line element $ds^2$ of Eq.(22) becomes asymptotically at spatial infinity

$$ds^2_{(as)} = \epsilon\Big([N^2_{(as)} - \vec{N}^2_{(as)}](d\tau)^2 - 2\vec{N}_{(as)} \cdot d\tau d\vec{\sigma} - d\vec{\sigma}^2\Big) + O(r^{-1}) =$$
$$= \epsilon\Big(\Big[\tilde{\lambda}^2_\tau - \vec{\tilde{\lambda}}^2 + (\tilde{\lambda}_\tau\tilde{\lambda}_{\tau s} - \tilde{\lambda}_r\tilde{\lambda}_{rs})\sigma^s + \frac{1}{4}(\tilde{\lambda}_{\tau u}\tilde{\lambda}_{\tau v} - \tilde{\lambda}_{ru}\tilde{\lambda}_{rv})\sigma^u\sigma^v\Big](d\tau)^2 +$$
$$+ 2(\tilde{\lambda}_r + \frac{1}{2}\tilde{\lambda}_{rs}\sigma^s) d\tau d\sigma^r - d\vec{\sigma}^2\Big) + O(r^{-1}) =$$
$$= \epsilon\Big([\tilde{\lambda}^2_\tau - \vec{\tilde{\lambda}} + 2(\tilde{\lambda}_\tau\frac{a_s}{c^2} + \epsilon_{sru}\tilde{\lambda}_r\frac{\omega^u}{c})\sigma^s + \frac{1}{c^2}(\frac{a^u a^v}{c^2} + \omega^u\omega^v - \delta^{uv}\vec{\omega}^2)\sigma^u\sigma^v](d\tau)^2 +$$

$$+ 2[\tilde{\lambda}_r - \epsilon_{rsu}\sigma^s \frac{\omega^u}{c}] d\tau d\sigma^r - d\vec{\sigma}^2\Big) + O(r^{-1}),$$

$$\tilde{\lambda}_{\tau r}(\tau) = 2\frac{a_r(\tau)}{c^2}, \quad acceleration,$$
$$\tilde{\lambda}_{rs}(\tau) = -2\epsilon_{rsu}\frac{\omega^u(\tau)}{c}, \quad angular\, velocity\, of\, rotation. \tag{27}$$

Since we have $\dot{x}^{(\mu)}_s(\tau) \stackrel{\circ}{=} b^{(\mu)}_{(\infty)A}\tilde{\lambda}^A(\tau)$, it follows that for $\tilde{\lambda}_\tau(\tau) = \epsilon$, $\tilde{\lambda}_r(\tau) = 0$, the origin moves with 4-velocity $(\epsilon; \vec{0})$ and has attached an accelerated rotating coordinate system [66]:

$$ds^2_{(as)} = {}^4\eta_{AB} d\sigma^A d\sigma^B + \frac{1}{c^2}[2\vec{a}\cdot\vec{\sigma} + (\frac{a^u a^v}{c^2} + \omega^u\omega^v - \delta^{uv}\vec{\omega}^2)\sigma^u\sigma^v](d\tau)^2 - 2\epsilon^{rsu}\sigma^s\frac{\omega^u}{c} d\tau d\sigma^u + O(r^{-1}),$$

which becomes inertial when $\tilde{\lambda}_{AB}(\tau) = 0$.

To go to the Wigner-like hypersurfaces [the analogue of the Minkowski Wigner hyperplanes with the asymptotic normal $l^{(\mu)}_{(\infty)} = l^{(\mu)}_{(\infty)\Sigma}$ parallel to $\hat{P}^{(\mu)}_{ADM}$ (i.e. $l^{(\mu)}_{(\infty)} = b^{(\mu)}_{(\infty)\tau} = \hat{P}^{(\mu)}_{ADM}/\sqrt{\epsilon\hat{P}^2_{ADM}}$); see Eqs.(21)] one follows the procedure defined for Minkowski spacetime:
i) one restricts oneself to spacetimes with $\epsilon p^2_{(\infty)} = {}^4\eta_{(\mu)(\nu)} p^{(\mu)}_{(\infty)} p^{(\nu)}_{(\infty)} > 0$ [this is possible, because the positivity theorems for the ADM energy imply that one has only timelike or light-like orbits of the asymptotic Poincaré group];
ii) one boosts at rest $b^{(\mu)}_{(\infty)A}(\tau)$ and $S^{(\mu)(\nu)}_{(\infty)}$ with the Wigner boost $L^{(\mu)}{}_{(\nu)}(p_{(\infty)}, \stackrel{\circ}{p}_{(\infty)})$;
iii) one adds the gauge-fixings $b^{(\mu)}_{(\infty)A}(\tau) \approx L^{(\mu)}{}_{(\nu)=A}(p_{(\infty)}, \stackrel{\circ}{p}_{(\infty)}) = \epsilon^{(\mu)}_A(u(p_{(\infty)}))$ [with $u^{(\mu)}(p_{(\infty)}) = p^{(\mu)}_{(\infty)}/\pm\sqrt{\epsilon p^2_{(\infty)}}$] and goes to Dirac brackets.
In this way one gets

$$S^{(\mu)(\nu)}_{(\infty)} \equiv \epsilon^{(\mu)}_C(u(p_{(\infty)}))\epsilon^{(\nu)}_D(u(p_{(\infty)}))\hat{J}^{CD}_{ADM} = S^{(\mu)(\nu)}_{ADM} \text{ and}$$
$$z^{(\mu)}_{(\infty)}(\tau, \vec{\sigma}) = x^{(\mu)}_{(\infty)}(\tau) + \epsilon^{(\mu)}_r(u(p_{(\infty)}))\sigma^r.$$

The origin $x^{(\mu)}_{(\infty)}$ is now replaced by the not covariant "external" center-of-mass canonical variable

$$\tilde{x}^{(\mu)}_{(\infty)} = x^{(\mu)}_{(\infty)} + \frac{1}{2}\epsilon^A_{(\nu)}(u(p_{(\infty)}))\eta_{AB}\frac{\partial\epsilon^B_{(\rho)}(u(p_{(\infty)}))}{\partial p_{(\infty)(\mu)}} S^{(\nu)(\rho)}_{(\infty)}$$

and one has

$$J^{(\mu)(\nu)}_{(\infty)} = \tilde{x}^{(\mu)}_{(\infty)} p^{(\nu)}_{(\infty)} - \tilde{x}^{(\nu)}_{(\infty)} p^{(\mu)}_{(\infty)} + \tilde{S}^{(\mu)(\nu)}_{(\infty)}$$

with $\tilde{S}^{(\mu)(\nu)}_{(\infty)} = S^{(\mu)(\nu)}_{(\infty)} - \frac{1}{2}\epsilon^A_{(\rho)}(u(p_{(\infty)}))\eta_{AB}(\frac{\partial\epsilon^B_{(\sigma)}(u(p_{(\infty)}))}{\partial p_{(\infty)(\mu)}} p^{(\nu)}_{(\infty)} - \frac{\partial\epsilon^B_{(\sigma)}(u(p_{(\infty)}))}{\partial p_{(\infty)(\nu)}} p^{(\mu)}_{(\infty)}) S^{(\rho)(\sigma)}_{(\infty)}$.

As in the Minkowski case one defines

$$\bar{S}^{AB}_{(\infty)} = \epsilon^A_{(\mu)}(u(p_{(\infty)}))\epsilon^B_{(\nu)}(u(p_{(\infty)}))\tilde{S}^{(\mu)(\nu)}_{(\infty)}$$

and one obtains at the level of Dirac brackets

$$
\begin{aligned}
\bar{S}^{\check{r}\check{s}}_{(\infty)} &\equiv \hat{J}^{\check{r}\check{s}}_{ADM},\\
\tilde{\lambda}_{AB}(\tau) &= 0,\\
-\tilde{\lambda}_A(\tau)\chi^A &= -\tilde{\lambda}_A(\tau)\epsilon^A_{(\mu)}(u(p_{(\infty)}))[p^{(\mu)}_{(\infty)} - \epsilon^{(\mu)}_B(u(p_{(\infty)}))\hat{P}^B_{AM}] =\\
&= -\tilde{\lambda}_A(\tau)\epsilon^A_{(\mu)}(u(p_{(\infty)}))[u^{(\mu)}(p_{(\infty)})(\epsilon_{(\infty)} - \hat{P}^\tau_{ADM}) - \epsilon^{(\mu)}_{\check{r}}(p_{(\infty)})\hat{P}^{\check{r}}_{ADM}] =\\
&= -\tilde{\lambda}_\tau(\tau)[\epsilon_{(\infty)} - \hat{P}^\tau_{ADM}] + \tilde{\lambda}_{\check{r}}(\tau)\hat{P}^{\check{r}}_{ADM},\\
\Rightarrow &\quad \epsilon_{(\infty)} - \hat{P}^\tau_{ADM} \approx 0, \qquad \hat{P}^{\check{r}}_{ADM} \approx 0.
\end{aligned}
\tag{28}
$$

in accord with Eqs.(21).

Therefore, on the Wigner-like hypersurfaces [they will be named Wigner-Sen-Witten hypersurfaces in the next Section and define the intrinsic asymptotic rest frame of the gravitational field; strictly speaking, the absence of supertranslations makes the scenario b) fully consistent only on these hypersurfaces], the remaining four extra constraints are:

$$
\begin{aligned}
&\hat{P}^{\check{r}}_{ADM} \approx 0,\\
&\epsilon_{(\infty)} = \sqrt{\epsilon p^2_{(\infty)}} \approx \hat{P}^\tau_{ADM} \approx M_{ADM} = \sqrt{\epsilon \hat{P}^2_{AM}}.
\end{aligned}
$$

Now the spatial indices have become spin-1 Wigner indices [they transform with Wigner rotations under asymptotic Lorentz transformations]. As said for parametrized theories in Minkowski spacetime, in this special gauge 3 degrees of freedom of the gravitational field become gauge variables, while $\tilde{x}^{(\mu)}_{(\infty)}$ becomes a decoupled observer with his clock near spatial infinity. These 3 degrees of freedom represent an "internal" center-of-mass 3-variable $\vec{\sigma}_{ADM}[{}^3g, {}^3\tilde{\Pi}]$ inside the Wigner-Sen-Witten hypersurface; $\sigma^{\check{r}} = \sigma^{\check{r}}_{ADM}$ is a variable representing the "center of mass" of the 3-metric of the slice $\Sigma_\tau$ of the asymptotically flat spacetime $M^4$ and is obtainable from the weak Poincaré charges with the group-theoretical methods of Ref. [58] as it is done in Ref. [59] for the Klein-Gordon field on the Wigner hyperplane. Due to $\hat{P}^r_{ADM} \approx 0$ we have

$$
\sigma^r_{ADM} \approx -\hat{J}^{\tau r}_{ADM}/\hat{P}^\tau_{ADM},
$$

so that $\vec{\sigma}_{ADM} \approx 0$ is equivalent to the requirement that the ADM boosts vanish.

When $\epsilon\hat{P}^2_{ADM} > 0$, with the asymptotic Poincaré Casimirs $\hat{P}^2_{ADM}$, $\hat{W}^2_{ADM}$ one can build the Möller radius $\rho_{AMD} = \sqrt{-\epsilon\hat{W}^2_{ADM}}/\hat{P}^2_{ADM}c$, which is an intrinsic classical unit of length like in parametrized Minkowski theories, to be used as an ultraviolet cutoff in a future attempt of quantization [67].

By going from $\tilde{x}^{(\mu)}_{(\infty)}$ [the non-covariant variable replacing $x^{(\mu)}_{(\infty)}$ after going to Dirac brackets with respect to the previous six pairs of second class constraints] and $p^{(\mu)}_{(\infty)}$ to the canonical basis [10]

$$
\begin{aligned}
&T_{(\infty)} = p_{(\infty)(\mu)} \tilde{x}^{(\mu)}_{(\infty)} / \epsilon_{(\infty)} = p_{(\infty)(\mu)} x^{(\mu)}_{(\infty)} / \epsilon_{(\infty)},\\
&\epsilon_{(\infty)},\\
&z^{(i)}_{(\infty)} = \epsilon_{(\infty)} (\tilde{x}^{(i)}_{(\infty)} - p^{(i)}_{(\infty)} \tilde{x}^{(o)}_{(\infty)} / p^{(o)}_{(\infty)}),\\
&k^{(i)}_{(\infty)} = p^{(i)}_{(\infty)} / \epsilon_{(\infty)} = u^{(i)}(p^{(\rho)}_{(\infty)}),
\end{aligned}
$$

one finds that the final reduction requires the gauge-fixings

$$T_{(\infty)} - \tau \approx 0 \text{ and } \sigma^{\check{r}}_{ADM} \approx 0.$$

Since $\{T_{(\infty)}, \epsilon_{(\infty)}\} = -\epsilon$, with the gauge fixing $T_{(\infty)} - \tau \approx 0$ one gets $\tilde{\lambda}_\tau(\tau) \approx \epsilon$, and the final Dirac Hamiltonian is

$$H_D = M_{ADM} + \tilde{\lambda}_{\check{r}}(\tau) \hat{P}^{\check{r}}_{ADM}, \qquad M_{ADM} \approx \hat{P}^\tau_{ADM}, \tag{29}$$

with $M_{ADM}$ [the ADM mass of the universe] the natural physical Hamiltonian to reintroduce an evolution in $T_{(\infty)} \equiv \tau$: namely in the rest-frame time identified with the parameter $\tau$ labelling the leaves $\Sigma_\tau$ of the foliation of $M^4$. See the Conclusions for comments on the problem of time in general relativity.

The final gauge fixings $\sigma^{\check{r}}_{ADM} \approx 0$ imply $\tilde{\lambda}_{\check{r}}(\tau) \approx 0$, $H_D = M_{ADM}$ and a reduced theory with the "external" center-of-mass variables $z^{(i)}_{(\infty)}$, $k^{(i)}_{(\infty)}$ decoupled [therefore the choice of the origin $x^{(\mu)}_{(\infty)}$ becomes irrelevant] and playing the role of a "point particle clock" for the time $T_{(\infty)} \equiv \tau$ [see the Conclusions]. There would be a weak form of Mach's principle, because only relative degrees of freedom would be present. That $M_{ADM}$ is the correct Hamiltonian for getting a $\tau$-evolution equivalent to Einstein's equations in spacetimes asymptotically flat at spatial infinity is also shown in Ref. [68].

The condition $\tilde{\lambda}_{AB}(\tau) = 0$ with $\tilde{\lambda}_\tau(\tau) = \epsilon$, $\tilde{\lambda}_r(\tau) = 0$ means that at spatial infinity there are no local (direction dependent) accelerations and/or rotations [$\vec{a} = \vec{\omega} = 0$]. The asymptotic line element is

$$
\begin{aligned}
&ds^2 = \epsilon\Big( [1 - \vec{\tilde{\lambda}}^2(\tau)] (d\tau)^2 + 2 \tilde{\lambda}_r(\tau) d\tau d\sigma^r - d(\vec{\sigma})^2 \Big) + O(r^{-1})\\
&= \epsilon\Big( {}^4\eta_{AB} d\sigma^A d\sigma^B - \vec{\tilde{\lambda}}^2(\tau)(d\tau)^2 + 2 \vec{\tilde{\lambda}}(\tau) \cdot d\tau d\vec{\sigma} \Big) + O(r^{-1}),
\end{aligned}
$$

which, for $\vec{\tilde{\lambda}}(\tau) = 0$ reduces to the line element of an inertial system near spatial infinity ["preferred asymptotic inertial observers"].

The asymptotic rest-frame instant form realization of the Poincaré generators becomes (no more reference to the boosts $\hat{J}^{\tau r}_{ADM}$)

$$
\begin{aligned}
&\epsilon_{(\infty)} = M_{ADM},\\
&p^{(i)}_{(\infty)},\\
&J^{(i)(j)}_{(\infty)} = \tilde{x}^{(i)}_{(\infty)} p^{(j)}_{(\infty)} - \tilde{x}^{(j)}_{(\infty)} p^{(i)}_{(\infty)} + \delta^{(i)\check{r}} \delta^{(j)\check{s}} \hat{J}^{\check{r}\check{s}}_{ADM},\\
&J^{(o)(i)}_{(\infty)} = p^{(i)}_{(\infty)} \tilde{x}^{(o)}_{(\infty)} - \sqrt{M^2_{ADM} + \vec{p}^2_{(\infty)}}\, \tilde{x}^{(i)}_{(\infty)} - \frac{\delta^{(i)\check{r}} \hat{J}^{\check{r}\check{s}}_{ADM} \delta^{(\check{s}(j)} p^{(j)}_{(\infty)}}{M_{ADM} + \sqrt{M^2_{ADM} + \vec{p}^2_{(\infty)}}}.
\end{aligned}
\tag{30}
$$

## IV. WIGNER-SEN-WITTEN 3-SURFACES.

In the previous Section the splitting from the lapse and shift functions of their asymptotic parts [corresponding to improper gauge transformations like in Yang-Mills theory] and the assumed form of these asymptotic parts can be interpreted as a restriction on the foliations realizing the 3+1 splittings of the spacetime $M^4$ [their leaves $\Sigma_\tau$ must tend to Minkowski spacelike hyperplanes at spatial infinity in a way independent from the direction]. This splitting of the lapse and shift functions is equivalent, inside parametrized Minkowski theories, to the gauge-fixings $z^{(\mu)}(\tau,\vec{\sigma}) - x^{(\mu)}(\tau) - b^{(\mu)}_r(\tau)\sigma^r \approx 0$, which restrict arbitrary spacelike hypersurfaces to spacelike hyperplanes. The non asymptotic part of lapse and shift functions $n(\tau,\vec{\sigma}) = m(\tau,\vec{\sigma})$, $n_r(\tau,\vec{\sigma}) = m_r(\tau,\vec{\sigma})$, [after having put $s(\tau,\vec{\sigma}) = s_{\check{r}}(\tau,\vec{\sigma}) = 0$ to kill supertranslations] are fields (vanishing at spatial infinity) on the hypersurfaces $\Sigma_\tau$ associated to these restricted foliations of $M^4$ [which can be called 'Minkowski-compatible'; they become foliations with spacelike hyperplanes of the Minkowski spacetime in rectangular coordinates when $G = 0$], which describe local deformations of these hypersurfaces.

Therefore, the boundary conditions defined in the previous Section and the connection with parametrized Minkowski theories restricted to spacelike hyperplanes, show that there exist special families of spacelike hypersurfaces $\Sigma_\tau$, diffeomorphic to $R^3$, in our class of asymptotically flat spacetimes $M^4$, which enjoy the same formal properties of spacelike hyperplanes in Minkowski spacetime [i.e. given an origin on each one of them and an adapted tetrad at this origin, there is a natural parallel transport so that one can uniquely define the adapted tetrads in all points of $\Sigma_\tau$ starting from the given adapted one at the origin] and which asymptotically agree with Minkowski spacelike hyperplanes. They correspond to the "non-flat preferred observers of Bergmann [3] (see also the Conclusions of II), namely there would be a set of "privileged observers" (privileged tetrads adapted to $\Sigma_\tau$) of "geometrical nature" (since they depend on the intrinsic and extrinsic geometry of $\Sigma_\tau$; on the solutions of Einstein's equations they also acquire a "dynamical nature" depending on the configuration of the gravitational field itself) and not of "static nature" like in the approaches of Möller [69], Pirani [70] and Goldberg [71]. These privileged observers are associated with the existence of the asymptotic Poincaré charges. A posteriori, namely after having solved Einstein's equations, one could try to use these "geometrical and dynamical" privileged observers (privileged non-holonomic coordinate systems replacing the rectangular Minkowski coordinates of the flat case) in the same way as, in metric gravity, are used the "bimetric theories", like the one of Rosen [72], with a set of privileged static non-flat background metrics. Since a congruence of timelike preferred observers means a tetrad field adapted to $\Sigma_\tau$, tetrad gravity is again preferred to metric gravity (the other reason being the fermion fields associated with matter). This congruence of timelike preferred observers[with asymptotic inertial observers when $\tilde{\lambda}_A(\tau) = (\epsilon; \vec{0})$ and $\tilde{\lambda}_{AB}(\tau) = 0$, see the end of previous Section] is a non-Machian element of these noncompact spacetimes. The asymptotic worldlines of the congruence may replace the static concept of "fixed stars" in the study of the precessional effects of gravitomagnetism on gyroscopes (dragging of inertial frames) and seem to be naturally connected with the definition of post-Newtonian coordinates (they require some concept of center of mass in their definition) [119].

Even if we do not have a characterization [like $z^{(\mu)}(\tau,\vec{\sigma}) = x^{(\mu)}(\tau) + b^{(\mu)}_r(\tau)\sigma^r$ for Minkowski hyperplanes] of the hypersurfaces $\Sigma_\tau$ corresponding to arbitrary spacelike hy-

perplanes in Minkowski spacetime and asymptotic to them in a direction-independent way at spatial infinity, we will see in this Section that the special families of hypersurfaces $\Sigma_\tau$ (denoted WSW, for Wigner-Sen-Witten, in the following), corresponding to the Wigner hyperplanes orthogonal to the 4-momentum of an isolated system and automatically selected by the requirement of absence of supertranslations, can be defined as those hypersurfaces having a certain rule of parallel transport and certain preferred adapted tetrad fields whose direction-independent value near spatial infinity is a tetrad whose timelike component is an angle-independent normal vector to $\Sigma_\tau$ at spatial infinity $l^{(\mu)}_{(\infty)} = l^{(\mu)}_{(\infty)\Sigma}$, [it is tangent to $S_\infty$], and is parallel to $\hat{P}^{(\mu)}_{ADM}$ (namely $\hat{P}^{(\mu)}_{ADM}$ is normal to $\Sigma_\tau$ at spatial infinity) as required by Eqs.(21) . Moreover, there are well defined equations determining these preferred adapted tetrads.

This picture can be obtained by putting together partial results of various authors. Ashtekar and Horowitz [73] pointed out the existence in metric gravity of a privileged family of lapse and shift functions, which can be extracted by the spinorial demonstration of Witten [74] of the positivity of the ADM energy (see Appendix C for a review of spinors on $M^4$ and on $\Sigma_\tau$). In our approach this family can be replaced by a different one determined by gauge-fixings implying $\tilde{\lambda}_{AB}(\tau) = 0$: this is the final condition of absence of supertranslations, see Eqs.(21). Then, Frauendiener [75] translated this fact in terms of privileged geometric adapted tetrads on each $\Sigma_\tau$ of this set of spacelike hypersurfaces, enjoying the same properties of tetrads adapted to Minkowski spacelike hyperplanes (he starts from the Sen-Witten equation [74,76–79] and uses ideas based on the Sparling 3-form [80,81]).

Let us review these arguments in more detail:

i) In his demonstration of the positivity energy theorem [i.e. $P_{ADM,(\mu)} n^{(\mu)} \geq 0$ for all future pointing null vectors ($n^2 = 0$), which implies $P_{ADM,(\mu)} n^{(\mu)} \geq 0$ for all future pointing asymptotic either timelike or null translations, with $n^{(\mu)} = lim_{r\rightarrow\infty}\, \sigma^{(\mu)}{}_{\tilde{A}\tilde{A}'} \xi^{\tilde{A}} \xi^{\tilde{A}'} \equiv \xi^{\tilde{A}} \xi^{\tilde{A}'}$ (by the usual identification of spinor and tensor indices $(\mu) \equiv \tilde{A}\tilde{A}'$) for some SU(2) spinor field on $\Sigma_\tau$], Witten [74] introduced SU(2) spinor fields on $\Sigma_\tau$ [see also Refs. [82,64]]. In the reformulation using the so called Nester-Witten 2-form $F(\xi)$ [82], defined on the total space of the spin bundle over $M^4$ as $F(\xi) = i\sigma^{\tilde{A}\tilde{A}'}_{(\mu)} \bar{\xi}_{\tilde{A}'}\, {}^4\nabla_{(\nu)} \xi_{\tilde{A}} dx^{(\mu)} \wedge dx^{(\nu)}$, one can show that $P_{ADM,(\mu)} n^{(\mu)} = lim_{r\rightarrow\infty} 2k \int_{\Sigma_\tau} F(\xi) = 2k \int_{S^2_{\tau,\infty}} dF(\xi)$. As first noted by Sparling [80] (see also the last chapter of Vol.2 of Ref. [79]) there is a 3-form $\Gamma$ on the spin bundle, the so called Sparling 3-form, such that

$$\Gamma = dF - \frac{1}{2} n^\mu\, {}^4G_{\mu\nu} X^\nu$$

$[X^\mu = \frac{1}{6} \epsilon^\mu{}_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma]$; therefore, the vacuum Einstein equations can be characterized by $d\Gamma = 0$. In presence of matter Einstein equations give $\Gamma \circeq dF - \frac{k}{2} n^\mu\, {}^4T_{\mu\nu} X^\nu$, so that $P_{ADM,(\mu)} n^{(\mu)} \circeq 2k \int_{S^2_{\tau,\infty}} (\Gamma + \frac{k}{2} n^\mu\, {}^4T_{\mu\nu} X^\nu)$. Using the dominant energy condition [83] for the positivity of the second term, one can arrive at the result $P_{ADM,(\mu)} n^{(\mu)} \geq 0$ if the SU(2) spinor $\xi^{\tilde{A}}$ $[n^{(\mu)} \equiv \xi^{\tilde{A}} \bar{\xi}^{\tilde{A}'}]$ satisfies the elliptic Sen-Witten equation for the noncompact hypersurface $\Sigma_\tau$ [see Eq.(C2)]

$$ {}^3\mathcal{D}_{\tilde A\tilde B}\psi^{\tilde B} = {}^3\tilde\nabla_{\tilde A\tilde B}\psi^{\tilde B} + \frac{1}{2\sqrt{2}}\,{}^3K\psi_{\tilde A} = 0. \qquad (31) $$

As stressed by Frauendiener and Mason [81], the Sparling 3-form is a Hamiltonian density for canonical general relativity (see also Ref. [84] on this point), while, when used quasi-locally, the 2-form F gives rise to Penrose's formula [85] for the angular momentum twistor of the quasi-local mass construction.

As further evidence that these ideas are required for treatment of conserved quantities in general relativity, it can be shown that the Sparling 3-form can be extended to be one of a collection of 3-forms "on the bundle of general linear frames" which, when pulled back to spacetime, give rise to classical formulae for the "pseudo-energy-momentum tensor" of the gravitational field [86] [see Ref. [87] for the Einstein complex, Ref. [62] for the Landau-Lifschitz one and Ref. [88] for a review]. See also Ref. [89], where the Sparling 3-form is studied in arbitrary dimension and where it is contrasted with Yang-Mills theory. In Ref. [90] there is the relationship of the Sparling 3-form to the spin coefficient formalism. These papers show the connection of the Poincaré charges with the standard theory of the Komar superpotentials and of the energy-momentum pseudotensors, which is reviewed in Appendix D.

See Refs. [91,73] for the existence of solutions of the Sen-Witten equation on noncompact spacelike hypersurfaces [for non-spacelike ones see the last chapter of Vol.2 in Ref. [79], its references and Ref. [92]]. See Refs. [93] for the non-unicity of Witten's positivity proof as first noted in Ref. [94]: other equations different from the Sen-Witten one can be used in variants of the proof.

In particular, in the paper of Reula in Ref. [91], used in Ref. [75], the problem of the existence of solutions of the Sen-Witten equation (31) has been formalized in the following way. An "initial data set" $(\Sigma_\tau, {}^3g_{rs}, {}^3K_{rs})$ for Einstein's equations consists of a 3-dimensional manifold $\Sigma_\tau$ without boundary equipped with a positive definite 3-metric ${}^3g_{rs}$ and a second rank, symmetric tensor field ${}^3K_{rs}$. For simplicity it is assumed that $\Sigma_\tau$ is diffeomorphic to $R^3$ and that ${}^3g_{rs}$ and ${}^3K_{rs}$ are $C^\infty$ (i.e. smooth) tensor fields on $\Sigma_\tau$. An initial data set is said to satisfy the "local energy condition" if the quantities

$\mu = \frac{1}{2}[{}^3R + {}^3K_{rs}\,{}^3K^{rs} - ({}^3K)^2]$ and $J_\mu = \partial^\nu[{}^3K_{\mu\nu} - {}^3g_{\mu\nu}\,{}^3K]$

[i.e. $[{}^3R - \frac{\epsilon}{2k\sqrt{\gamma}}\tilde{\mathcal{H}}](\tau,\vec\sigma)$ and $[-\{\frac{1}{2}{}^3\tilde{\mathcal{H}}^r + {}^3\Gamma^r_{su}\,{}^3\tilde\Pi^{su} + {}^3\tilde\Pi^{rs}\partial_s ln\sqrt{\gamma}\}/\sqrt{\gamma}](\tau,\vec\sigma)$ in the ADM canonical metric gravity formalism (Section V of I)] satisfy

$\mu \geq \mid J^\mu J_\mu \mid^{1/2}$.

An initial data set is "asymptotically flat" if one can introduce an asymptotically Euclidean coordinate system such that ${}^3g_{rs} - \delta_{rs} = O(r^{-1})$ and $\partial_u\,{}^3g_{rs} = O(r^{-2})$ for $r \to \infty$ and, moreover, ${}^3K_{rs} = O(r^{-2})$ and ${}^3R_{rs} = O(r^{-3})$ for $r \to \infty$ [they are compatible with Christodoulou-Klainermann Eqs.(1)]. Then one has the following existence theorem(see also Ref. [73]):

If $(\Sigma_\tau, {}^3g_{rs}, {}^3K_{rs})$ is an initial data set that satisfies the local energy condition and is asymptotically flat, then for any spinor field $\psi_o^{\tilde A}$ that is "asymptotically constant" (i.e. $\partial_r\psi_o^{\tilde A} = 0$

outside a compact subset of $\Sigma_\tau$; see also Ref. [92]) there exists a spinor field $\psi^{\tilde A}$ satisfying the Sen-Witten equation (31) and such that $\psi^{\tilde A} = \psi_o^{\tilde A} + O(r^{-1})$ at spatial infinity.

ii) In Ref. [73], Ashtekar and Horowitz note that the Sen-Witten equation enables one to transport rigidly constant spinors at infinity to the interior of the 3-manifold on which the initial data are defined. By "taking squares" of the Sen-Witten spinors one can construct a "preferred" family of lapse and shifts and interpret them as the projections of 4-dimensional null evolution vector fields $z^\mu_\tau(\tau,\vec\sigma) = [N l^\mu + N^\mu](\tau,\vec\sigma)$, $N^\mu(\tau,\vec\sigma) = [z^\mu_r N^r](\tau,\vec\sigma)$, $[l^\mu N_\mu](\tau,\vec\sigma) = 0$, $z^2_\tau(\tau,\vec\sigma) = 0$, obtained by transporting rigidly the spacetime asymptotic translations at spatial infinity. The preferred family correspond to a "gauge fixing prescription" for lapse and shift functions. Next it is shown that, on the phase space of general relativity, one can compute Hamiltonians corresponding to these lapse and shifts. Although these Hamiltonians have a complicated form in terms of the usual canonical variables (involving volume and surface integrals), they are simply the volume integrals of squares of derivatives of the Witten spinors. In particular, the Hamiltonians generating Witten-time translations are manifestly positive and differentiable.

To define the preferred 4-parameter family of lapses and shifts, they proceed as follow. Since Sen-Witten's equation enables one to "transport" constant spinors $\psi_o^{\tilde A}$ at infinity to the "interior" of $\Sigma_\tau$ [the rotation of the spinor at infinity causes a rigid rotation on the entire spinor field], consequently, there is available on $\Sigma_\tau$ a "distinguished" complex 2-dimensional vector space of asymptotically constant spinor fields $\psi^{\tilde A}$. With each of these spinor fields $\psi^{\tilde A}$ we shall associate a lapse-shift pair $(N, N^\mu)$ given by $N \equiv \psi^{+\tilde A}\psi_{\tilde A}$ and $N^\mu \equiv -i\sqrt{2}\,\psi^{+(\tilde A}\psi^{\tilde B)}$. Let $\alpha^{\tilde A}$ and $\beta^{\tilde A}$ be two linearly independent Witten transported spinor fields. Then, by consecutively substituting $\alpha^{\tilde A}$, $\beta^{\tilde A}$, $(\alpha^{\tilde A} + \beta^{\tilde A})$ and $(\alpha^{\tilde A} + i\beta^{\tilde A})$ for $\psi^{\tilde A}$ in the above prescription, one obtains 4 pairs $(N_{(k)}, N^\mu_{(k)})$ with $(k) = 1, 2, 3, 4$, of lapse-shifts pairs each of which defines an asymptotic null translation. Consider the real 4-dimensional vector space generated by these pairs: this space is independent of the initial choice of $\alpha^{\tilde A}$ and $\beta^{\tilde A}$. This is the preferred family of lapses and shifts. Each element of this family defines an asymptotic translation and is, in turn, determined by this translation.

These expressions are essentially spinorial, i.e. they depend on the phases of the individual spinors whereas the original lapse-shift vector did not . It is essential for a coherent point of view, therefore, to regard the spinors as fundamental, and the lapse-shift vector as derived (this requires supergravity, which motivated Witten, but is not justified in ordinary gravity). The Witten argument required that the phases of the spinors making up the null lapse-shift vector be assumed to be asymptotically constant along with the lapse-shift vector: without this, the argument fails.

In terms of vectors, given a "tetrad" at infinity, it is noted in Ref. [73]that the SL(2,C) Sen-Witten equation then provides us with a tetrad field everywhere on $\Sigma_\tau$. If we rotate the tetrad at infinity, the entire field rotates rigidly by the same amount; the freedom is that of global rather than local Lorentz transformations. It is in this sense that we have a "gauge fixation procedure". Note, however, that the preferred tetrad fields depend on the choice of the variables $({}^3g_{rs}, {}^3K_{rs})$ on $\Sigma_\tau$; if we change the metric ${}^3g_{rs}$ near $\Sigma_\tau$, the tetrad fields change.

It can also be shown [73] that if ${}^3T^\mu$ is a vector field tangent to $\Sigma_\tau$ (not necessarily spacelike) with asymptotic value ${}^3T^\mu_{(\infty)}$, then ${}^3T^\mu$ is timelike (respectively, null, spacelike)

everywhere, if ${}^3T^{\mu}_{(\infty)}$ is timelike (respectively, null, spacelike) at infinity.

Then, in Ref. [73] it is noted that, if $(M^4, {}^4\eta_{(\mu)(\nu)})$ is the Minkowski spacetime, then, since the constant spinor fields in it automatically satisfy Sen-Witten equation, for any choice of $\Sigma_\tau$, the transport of translations at infinity yields the translational Killing fields everywhere on $M^4$. In a generic spacetime, however, the transport is tied to the choice of $\Sigma_\tau$. Thus, it is only when we are given a foliation of a generic spacetime that we can obtain 4 vector fields everywhere on the spacetime, and they "depend" on the choice of the foliation. The transport is well suited to the canonical framework, however, because in this framework one deals only with 3-surfaces.

iii) All these results can be rephrased in our language, by noting that the family of preferred lapse and shift functions of Ref. [73] could be replaced with the family which can be obtained from Eq.(20) when $\tilde{\lambda}_{AB}(\tau) = 0$ so that $N_{(as)A}(\tau, \vec{\sigma}) = N_{(as)A}(\tau) = -\tilde{\lambda}_A(\tau)$. Our 4 arbitrary functions $\tilde{\lambda}_A(\tau)$ give the same multiplicity as in the previous spinorial construction without relying on the special null evolution vectors needed in it. Therefore, in our approach, the "gauge-fixing prescription" for selecting the preferred family of lapse and shifts becomes the requirement of absence of supertranslations according to Eqs.(21), i.e. $\tilde{\lambda}_{AB}(\tau) = 0$. But this implies $\hat{P}^{(\mu)}_{ADM} \approx l^{(\mu)}_{(\infty)} \hat{P}^{\tau}_{ADM}$ and, as a consequence, the allowed foliations and their leaves, i.e. the spacelike hypersurfaces $\Sigma_\tau$, could be called "Wigner-Sen-Witten" (WSW) foliations and spacelike hypersurfaces, being the analogues of the Wigner foliations and spacelike hyperplanes of the parametrized Minkowski theories.

iv) The final step, also to justify our change of the family of lapse and shift functions, is to eliminate completely any reference to spinors and to reformulate the properties of these WSW spacelike hypersurfaces $\Sigma_\tau$ only in terms of triads on $\Sigma_\tau$ and adapted tetrads (see Section III of I for the transition to general tetrads). This has been done in Ref. [75], where Frauendiener , exploiting the fact that there is a unique (up to a global sign) correspondence between a spinor and a triad on a spacelike hypersurface, derives the necessary and sufficient conditions that have to be satisfied by a triad in order to correspond to a spinor that satisfies the Sen-Witten equation.

Given a SU(2) spinor $\psi^{\tilde{A}}$, one constructs the symmetric object $\psi^{\tilde{A}}\psi^{\tilde{B}}$, which corresponds to a spatial 3-vector ${}^3m^{\mu}$ (instead, a SL(2,C) spinor corresponds to a selfdual bivector [79]); this vector is complex and null: ${}^3m_\mu\, {}^3m^\mu = 0$. Conversely, every complex spatial null vector defines a SU(2) spinor $\psi^{\tilde{A}}$ up to a sign. The spinor $\sqrt{2}\psi^{\dagger\, (\tilde{A}}\, \psi^{\tilde{B})}$ is real and corresponds to a 3-vector ${}^3u^\mu$ orthogonal to ${}^3m_\mu$, ${}^3m_\mu\, {}^3u^\mu = 0$. Writing ${}^3m^\mu = \frac{1}{\sqrt{2}}({}^3x^\mu + i\, {}^3y^\mu)$ and defining $a = \psi^{\dagger}_{\tilde{A}}\, \psi^{\tilde{A}}$, one gets ${}^3x^\mu\, {}^3y_\mu = {}^3x^\mu\, {}^3u_\mu = {}^3y^\mu\, {}^3u_\mu = 0$, ${}^3x^\mu\, {}^3x_\mu = {}^3y^\mu\, {}^3y_\mu = {}^3u^\mu\, {}^3u_\mu = -\epsilon a^2$. Therefore, going to the holonomic basis of $\Sigma_\tau$, one has ${}^3u^r = -\frac{1}{a}\epsilon^{ruv}\, {}^3x^u\, {}^3y^v$ and one can define an oriented triad ${}^3e^{(W)r}_{(a)}$ on $\Sigma_\tau$: ${}^3e^{(W)r}_{(1)} = {}^3x^r$, ${}^3e^{(W)r}_{(2)} = {}^3y^r$, ${}^3e^{(W)r}_{(3)} = {}^3u^r$. Since the SU(2) spinor $i\sqrt{2}\partial_{(\tilde{A}}{}^{\tilde{B}}\, \psi_{\tilde{C})\, \tilde{B}}$ corresponds to the curl $\epsilon^{ruv}\partial^u\, {}^3v^v$ of a 3-vector ${}^3v^r$, one can use the Sen-Witten equation for $\psi^{\tilde{A}}$ to get an equation for $v_{\tilde{A}\tilde{B}} = \psi_{\tilde{A}}\psi_{\tilde{B}} \equiv {}^3m_\mu$. Since the Sen-Witten equation corresponds to 4 real equations for 4 real unknown, two of the 6 equations for ${}^3m_\mu$ cannot be independent and generate constraints on the triad. The final result is that to each solution $\psi^{\tilde{A}}$ of the Sen-Witten equation it corresponds a triad on the WSW hypersurface $\Sigma_\tau$, satisfying a certain cyclic condition, with two divergence free 3-vectors

and with the third one having a non-vanishing divergence proportional to the trace of the extrinsic curvature of $\Sigma_\tau$ [on a maximal WSW hypersurface (${}^3K=0$) all three vectors are divergence free ]

$$
\begin{aligned}
&{}^3\nabla_r\, {}^3e^{(W)r}_{(1)} = {}^3\nabla_r\, {}^3e^{(W)r}_{(2)} = 0,\\
&{}^3\nabla_r\, {}^3e^{(W)r}_{(3)} = -\alpha {}^3K,\\
&{}^3e^{(W)r}_{(1)}\, {}^3e^{(W)s}_{(3)}\, {}^3\nabla_r\, {}^3e^{(W)}_{(2)s} + {}^3e^{(W)r}_{(3)}\, {}^3e^{(W)s}_{(2)}\, {}^3\nabla_r\, {}^3e^{(W)}_{(1)s} + {}^3e^{(W)r}_{(2)}\, {}^3e^{(W)s}_{(1)}\, {}^3\nabla_r\, {}^3e^{(W)}_{(3)s} = 0.
\end{aligned}
\tag{32}
$$

There is a 2-1 correspondence between solutions to the Sen-Witten equation and such triad fields on the WSW $\Sigma_\tau$. It may be checked that the 4-dimensional freedom in the choice of a spinor at one point (at spatial infinity) implies that a triad satisfying these conditions is unique up to global frame “rotations” and “homotheties”. In this sense, the geometry of an initial data set “uniquely” determines a “triad” and hence together with the associated surface normal [$l^{(\mu)}_{(\infty)} = l^{(\mu)}_{(\infty)\Sigma}$ parallel to $\hat{P}^{(\mu)}_{ADM}$ at spatial infinity] an “adapted tetrad” ${}^4_{(\Sigma)}\check{E}^{(W)(\mu)}_A$ in spacetime according to Eq.(40) of I [with $N, N^r$ given by Eq.(20) with $\tilde{\lambda}_{AB}(\tau)=0$]. We may call these triads “geometrical triads”. In Ref. [75] it is shown: 1) these triads do not exist for compact $\Sigma_\tau$; 2) with nontrivial topology for $\Sigma_\tau$ there can be less than 4 real solutions and the triads cannot be build; 3) the triads exist for asymptotically null surfaces, but the corresponding tetrad will be degenerate in the limit of null infinity.

Moreover, in Ref, [75], using the results of Ref. [86], it is noted that the Einstein energy-momentum pseudo-tensor [87] is a canonical object only in the frame bundle over $M^4$, where it coincides with the Sparling 3-form. In order to bring this 3-form back to a 3-form (and then to an energy-momentum tensor) over the spacetime $M^4$, one needs a section (i.e. a tetrad) in the frame bundle. Only with the 3+1 decomposition of $M^4$ with WSW foliations one gets that (after imposition of Einstein's equations together with the local energy condition) one has a preferred (geometrical and dynamical) adapted tetrad on the initial surface $\Sigma_\tau$.

v) Then one has the geometric problem of determining the Wigner-Sen-Witten 3-hypersurfaces in $M^4$ given a solution of the Hamilton equations. With a set of cotriads ${}^3\hat{e}_{(a)r}$ solution of the equation of motion we can construct the associated extrinsic curvature ${}^3K_{rs}$. Then Eqs.(32) allow to find the associated triads ${}^3e^{(W)r}_{(a)}$ and we can define the $\Sigma_\tau$-adapted tetrads (see Section III of I) such that the asymptotic normal $l^{(\mu)}_{(\infty)}$ is parallel to the weak ADM 4-momentum associated with the solution ${}^3\hat{e}_{(a)r}$.

The Wigner-Sen-Witten hypersurfaces are those surfaces admitting the given adapted tetrad fields. By using Eq.(40) of I, we can obtain the $\Sigma_\tau$-adapted cotetrads ${}^4_{(\Sigma)}\check{\check{E}}^{(\alpha)}_A(\tau,\vec{\sigma})$ associated with a solution of the Hamilton equations ${}^3e_{(a)r}$, $N$, $N_{(a)}$; then, with the transition coefficients $b^A_\mu(\tau,\vec{\sigma}) = \partial\sigma^A(z)/\partial z^\mu$, we obtain the $\Sigma_\tau$- adapted cotetrads ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\tau,\vec{\sigma}))$ of Eqs.(39) of I. Since $M^4$ is a globally hyperbolic spacetime, we have ${}^4_{(\Sigma)}\check{E}^{(o)}_\mu(z(\tau,\vec{\sigma})) = \epsilon l_\mu(z(\tau,\vec{\sigma})) = \epsilon N(\tau,\vec{\sigma})\partial_\mu\tau(z)$ [$l^\mu(\tau,\vec{\sigma})$ is the normal to $\Sigma_\tau$ in $z^\mu(\tau,\vec{\sigma})$]. Therefore, from the equation $\partial_\mu\tau(z) = l_\mu(z)/N(z)$ we can determine the function $\tau(z)$ associated with the given solution. The WSW hypersurface $\Sigma_\tau$ associated with the given solution is the set of points $z^\mu(\tau,\vec{\sigma})$ such that $\tau(z)=\tau$. This allows to find the functions $z^\mu_{wsw}(\tau,\vec{\sigma})$ defining the embedding of the Wigner-Sen-Witten hypersurfaces in $M^4$ and giving its

Wigner-Sen-Witten foliations: by construction they satisfy [on WSW hypersurfaces we have $b^{(\mu)}_{(\infty)r}(\tau) = \epsilon^{(\mu)}_r(u(p_{(\infty)}))\sigma^r$]

$$z^\mu_{wsw}(\tau,\vec{\sigma}) \rightarrow_{|\vec{\sigma}|\rightarrow\infty} \delta^\mu_{(\mu)} z^{(\mu)}_{(\infty)}(\tau,\vec{\sigma}) = \delta^\mu_{(\mu)}[x^{(\mu)}_{(\infty)}(\tau) + \epsilon^{(\mu)}_r(u(p_{(\infty)}))\sigma^r]$$

with $x^{(\mu)}_{(\infty)}(0)$ arbitrary [it reflects the arbitrariness of the absolute location of the origin of asymptotic coordinates (and, therefore, also of the "external" center of mass $\tilde{x}^{(\mu)}_{(\infty)}(0)$) near spatial infinity]. See Ref. [95] and its interpretation of the center of mass in general relativity (this paper contains the main references on the problem starting from Dixon's definition [61]): $x^{(\mu)}_{(\infty)}(\tau)$ may be interpreted as the arbitrary "reference" (or "central") timelike worldline of this paper.

This also allows the determination of the coefficients $b^\mu_A(\tau,\vec{\sigma}) = \partial z^\mu(\tau,\vec{\sigma})/\partial\sigma^A$ allowing the transition from general 4-coordinates to adapted 4-coordinates. Since ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu dz^\mu = {}^4_{(\Sigma)}\theta^{(\alpha)}$ are non-holonomic coframes (see Appendix A of II), there are not coordinate hypersurfaces and lines for the associated non-holonomic coordinates $z^{(\alpha)}$ [98] on $M^4$; as shown in Ref. [99] for them we have ${}^4_{(\Sigma)}\theta^{(\alpha)} = dz^{(\alpha)} + z^{(\beta)}\Big[{}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu \frac{\partial\, {}^4_{(\Sigma)}\check{E}^\mu_{(\beta)}}{\partial z^{(\gamma)}}\Big] dz^{(\gamma)}$.

Let us remark that in void spacetimes (see Section VI) one has $\hat{P}^{(\mu)}_{ADM} = 0$ and WSW hypersurfaces do not exist in them: therefore it is only in presence of matter that we can recover the Wigner hyperplane underlying a given Wigner-Sen-Witten hypersurface. Since in parametrized Minkowski theories Wigner hyperplanes are orthogonal to the total 4-momentum of the isolated matter-field system, such hyperplanes are not defined in absence of fields and matter.

In conclusion it turns out that with WSW Minkowski-compatible foliations with spacelike hypersurfaces $\Sigma_\tau$ , preferred adapted tetrads and cotetrads are associated. Therefore, there are "preferred geometrical observers" associated with the leaves $\Sigma_\tau$ of a WSW foliation, which are determined by both the intrinsic and extrinsic (${}^3K$) geometry of these $\Sigma_\tau$'s.

It is not clear whether there exists a characterization of the more general foliations with $\tilde{\lambda}_{AB}(\tau) \neq 0$, which have associated unavoidable supertranslations and have ill-defined $\Sigma_\tau$-adapted tetrads at spatial infinity due to the terms linear in $\vec{\sigma}$ in the lapse and shift functions.

Let us finish this Section by quoting the formulation of general relativity as a "teleparallel" theory done by Nester in Refs. [96] in order to prove the positivity of gravitational energy with purely tensorial methods. It could be connected either with a different notion of parallel transport on the WSW hypersurfaces or with the characterization of the Minkowski-compatible hypersurfaces $\Sigma_\tau$ corresponding to arbitrary Minkowski hyperplanes [$l^\mu_{(\infty)} = l^{(\mu)}_{(\infty)\Sigma}$ not parallel to $P^{(\mu)}_{ADM}$] and not to Wigner hyperplanes.

Nester shows that by imposing certain gauge conditions on tetrads one can obtain positivity of the ADM energy. His conditions are closely related to Eqs.(32). Specifically, he also imposes the cyclic condition but on global cotriads rather than on global triads. Clearly, a global triad defines a connection on an initial surface, by requiring that a parallel vector field has constant coefficients with respect to the triad. This connection will be metric compatible and integrable since it preserves the triad. Therefore, its curvature will be zero, but the torsion will be nonzero. We see from the present result, that on an initial data set satisfying the local energy conditions (needed to prove the existence of Sen-Witten spinors)

there exists a "preferred absolute parallelism".

While the orthonormal coframe ${}^3\theta^{(a)} = {}^3e^{(a)}_r d\sigma^r = {}^3e_{(a)r} d\sigma^r$ determines the metric and the Riemannian geometry, a given Riemannian geometry determines only an equivalence class of orthonormal coframes: coframes are defined only modulo position-dependent rotations and, under these gauge transformations, the spin connection transforms as a SO(3) gauge potential (see Section III of I). A gauge-fixing for the rotation freedom usually means a choice of a representative of the spin connection inside its gauge orbit [like the Coulomb gauge for the electromagnetic vector gauge potential $\vec{A}$]: this would induce a choice of an associated coframe with respect to some standard origin. However, since coframes ${}^3\theta^{(a)}$ are more elementary of the Levi-Civita spin connection ${}^3\omega^{(a)}{}_{(b)}$ [which is built in terms of them], it is possible to define gauge-fixings directly at the level of coframes [see Ref. [96],papers b)]. The idea of these papers is that the choice of a preferred coframe ${}^3\theta^{(a)}_{(P)}$ on the Riemannian parallelizable 3-manifold $(\Sigma_\tau, {}^3g)$ [with its associated metric compatible Levi-Civita connection and parallel transport and vanishing torsion] may be associated with the definition of a new kind of parallel transport on $\Sigma_\tau$, i.e. of a "teleparallel" (or "weitzenböck or distant parallelism") geometry on $\Sigma_\tau$, according to which a covariant vector is parallely transported along a curve in $\Sigma_\tau$ if in each point q of the curve it has the same components with respect to the local coframe ${}^3\theta^{(a)}_{(P)}|_q$. The special coframe ${}^3\theta^{(a)}_{(P)}$ is said "orthoteleparallel" (OT) coframe. With this structure $(\Sigma_\tau, \delta_{(a)(b)})$ is a 3-manifold with flat metric [the curvature vanish because this parallel transport is path-independent (absolute parallelism) like in Euclidean geometry] and the OT coframe ${}^3\theta^{(a)}_{(P)}$ is that special coframe in which by construction also all the spin connection coefficients vanish], but with a nonvanishing "torsion" [it completely characterizes this kind of geometry]

$$
\begin{aligned}
&{}^3T^{(a)}_{(P)} = d\,{}^3\theta^{(a)}_{(P)} = -\tfrac{1}{2}\,{}^3C^{(a)}{}_{(b)(c)}\,{}^3\theta^{(b)}_{(P)} \wedge {}^3\theta^{(c)}_{(P)},\\
&{}^3C^{(a)}{}_{(b)(c)} = -d\,{}^3\theta^{(a)}({}^3e_{(b)}, {}^3e_{(c)}),\ {}^3e_{(a)} = {}^3e^r_{(a)}\partial/\partial\sigma^r.
\end{aligned}
$$

The Riemannian geometry $(\Sigma_\tau, {}^3g)$ corresponds to a whole equivalence class of teleparallel geometries $(\Sigma_\tau, {}^3\theta^{(a)}_{(P)})$, according to which coframe is chosen as the preferred OT one.

In Ref. [96]b) it is pointed out that there exists a natural (of elliptic type) gauge-fixing for the choice of a special OT coframe ${}^3\theta^{(a)}_{(P)}$ [${}^3e_{(P)(a)}$ is the dual OT frame]:

$$
\begin{aligned}
&\delta\hat{q}_{(P)} = 0 \quad \Rightarrow \quad d{*}\hat{q}_{(P)} = 0\\
&d\tilde{q}_{(P)} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{q}_{(P)} &= i_{{}^3e_{(P)(a)}}\,{}^3T^{(a)}_{(P)} = i_{{}^3e_{(P)(a)}}\,d\,{}^3\theta^{(a)}_{(P)} = -{}^3C^{(a)}{}_{(a)(b)}\,{}^3\theta^{(b)},\\
&(this\, 1-form\, is\, the\, algebraically\, irreducible\, trace\, of\, the\, teleparralel\, torsion),
\end{aligned}
$$

$$
\begin{aligned}
\hat{q}_{(P)} &= \frac{1}{2}{}^3\theta^{(a)}_{(P)} \wedge \delta_{(a)(b)}\,{}^3T^{(b)}_{(P)} = \frac{1}{4}{}^3C_{(a)(b)(c)}\,{}^2\theta^{(a)}_{(P)} \wedge {}^3\theta^{(b)}_{(P)} \wedge {}^3\theta^{(c)}_{(P)},\\
&(this\, 3-form\, is\, the\, totally\, antisymmetric\, part\, of\, the\, teleparallel\, torsion),
\end{aligned}
$$

$$\Downarrow$$

$$-{}^3C^{(a)}{}_{(b)(c)} = \frac{2}{3}(q^{(a)}_{(P)(b)(c)} - q^{(a)}_{(P)(c)(b)}) + \frac{1}{2}(\delta^{(a)}_{(b)}\tilde{q}_{(P)(c)} - \delta^{(a)}_{(c)}\tilde{q}_{(P)(b)}) + \frac{2}{3}\hat{q}^{(a)}_{(P)(b)(c)},$$

$${}^3\Gamma^{(a)(b)}_{(P)}{}_{(c)} = \frac{2}{3}(q_{(P)(c)}{}^{(b)(a)} - q_{(P)(c)}{}^{(a)(b)}) - \frac{1}{2}(\delta^{(a)}_{(c)}\tilde{q}^{(b)}_{(P)} - \delta^{(b)}_{(c)}\tilde{q}^{(a)}_{(P)}) + \frac{1}{3}\hat{q}^{(a)(b)}_{(P)}{}_{(c)},$$

$$where\, q_{(P)(a)(b)(c)} \quad is \quad -\frac{1}{2}[{}^3C_{(a)(b)(c)} + {}^3C_{(b)(a)(c)}], \qquad with\, all\, traces\, removed. \tag{33}$$

Here $\delta$ is the codifferential [$\delta = *d*$ with $*$ the Hodge dual; $\epsilon_{123} = 1$; for noncompact $\Sigma_\tau$ suitable boundary conditions are needed]; $*\hat{q}_{(P)}$ is a function and $*\delta\hat{q}_{(P)}$ a 1-form.

These are three conditions (one is the cyclic condition), which determine a special orthonormal coframe on a 3-manifold [i.e. they determine the 3 Euler angles of the dual frame with respect to a standard frame chosen as an identity cross section in the orthonormal frame bundle $F(\Sigma_\tau)$] once appropriate boundary conditions are fixed. For asymptotically flat 3-manifolds [${}^3g \rightarrow \delta + O(1/r)$] the boundary condition is $\tilde{q}_{(P)} \rightarrow 0$ for $r = \sqrt{\vec{x}^2} \rightarrow \infty$ and $*\hat{q}_{(P)} = 0$. When the first de Rahm cohomology group $H_1(\Sigma_\tau) = 0$ vanishes, the closed 1-form $\tilde{q}_{(P)}$ is globally exact in this gauge, $\tilde{q}_{(P)} = dF_{(P)}$, and determines a function $F_{(P)}$ up to a constant, which may be suitably normalized at infinity, and it can be shown [96]c), that it is the best definition of the generalization of the Newton potential: it is the scale factor which satisfies the superhamiltonian constraint equation. With this gauge [96]c), one gets a locally positive representation for the Hamiltonian density allowing a new, "strictly tensorial" (in contrast to Witten's spinor method [74]) proof of positive energy for Einstein's theory of gravity.

Given an orthonormal coframe ${}^3\theta^{(a)}$, the gauge conditions (33) become a nonlinear second-order elliptic system for the rotation matrix defining an OT coframe ${}^3\theta^{(a)}_{(P)} = R^{(a)}{}_{(b)}{}^3\theta^{(b)}$. In Ref. [96]b) it is shown that the associated linearized problem has a unique solution if $d^3\theta^{(a)}$ is not too large and the second deRahm cohomology group $H_2(\Sigma_\tau) = 0$ vanishes [for asymptotically flat spaces one should use the first paper in Ref. [91]]. In Ref. [97] it is shown that for 3-manifolds the gauge conditions (33) are essentially equivalent to the "linear" Dirac equation, for which unique solutions exist. Hence for 3-manifolds special OT coframes exist except possibly at those (isolated) points where the Dirac spinor vanishes.

## V. POINCARÉ CHARGES AND THE PHYSICAL HAMILTONIAN IN TETRAD GRAVITY.

In the formulation of tetrad gravity given in I and II we used the ADM action. Therefore, all the discussion of Section III about the differentiability of the Hamiltonian, the definition of Poisson brackets, the definition of proper and improrper gauge transformations can be directly reformulated in tetrad gravity. The only difference inside tetrad gravity in the Hamiltonian treatment of quantities depending upon ${}^3g_{rs}(\tau,\vec{\sigma})$, ${}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma})$, is that now we have $\{{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma}^{\prime})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{\prime}) F^{rsuv}_{(a)(b)}(\tau,\vec{\sigma})\, {}^3\tilde{M}_{(a)(b)}(\tau,\vec{\sigma}) \approx 0$ [see Eqs.(84) of I] and not $= 0$. Therefore, constants of motion (functional $F[{}^3g_{rs}, {}^3\tilde{\Pi}^{rs}]$) of metric gravity remain such in tetrad gravity, since they have weakly zero Poisson brackets with $\tilde{\mathcal{H}}(\tau,\vec{\sigma})$, ${}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma})$ [and, therefore, with ${}^3\tilde{\Theta}_r(\tau,\vec{\sigma})$ and $\hat{\tilde{\mathcal{H}}}_{(a)}(\tau,\vec{\sigma})$, see Eqs.(61), (62), (79) and (85) of I] and also with the other first class constraints $\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$ [see Eqs. (54) and (55) of I].

As a consequence the weak and strong Poincaré charges are still constants of motion in tetrad gravity and their weak Poincaré algebra under Poisson brackets may only be modified by extra terms containing ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$. A more complete study of these properties would require the study of the quasi-invariances of the Lagrangian density of tetrad gravity [Eq.(51) of I] under the gauge transformations generated by the 14 first class constraints of the theory [using the second Noether theorem as it was done in Appendix A for metric gravity].

The only lacking ingredients are the definition of proper gauge transformations generated by the primary (without associated secondary) first class constraints $\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) \approx 0$, and the boundary conditions for cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$, because the lapse and shift functions $N(\tau,\vec{\sigma})$, $N_{(a)}(\tau,\vec{\sigma}) = {}^3e^r_{(a)}(\tau,\vec{\sigma}) N_r(\tau,\vec{\sigma})$ are treated in the same way as in metric gravity.

Therefore, we shall assume that there exist the same coordinate systems of $M^4$ and $\Sigma_\tau$ as in metric gravity [for the sake of simplicity the indices $\check{r}$ are replaced with $r$] and that the $\Sigma_\tau$ -adapted tetrads of Eqs.(39) of I, whose expression is ${}^4_{(\Sigma)}\check{E}^\mu_{(\mu)}$ with

$$
{}^4_{(\Sigma)}\check{E}^\mu_{(o)} = l^\mu \qquad , {}^4_{(\Sigma)}\check{E}^\mu_{(a)} = {}^3e^s_{(a)} b^\mu_s
$$

[$b^\mu_A$ are the transformation coefficients to $\Sigma_\tau$ -adapted coordinates], have a well defined angle-independent limit ${}^4_{(\Sigma)}\check{E}^\mu_{(\infty)(\mu)}$ at spatial infinity, such that

$$
{}^4_{(\Sigma)}\check{E}^\mu_{(\infty)(o)} = \delta^\mu_{(\mu)} l^{(\mu)}_{(\infty)} = \delta^\mu_{(\mu)} b^{(\mu)}_{(\infty)\tau} \qquad , {}^4_{(\Sigma)}\check{E}^\mu_{(\infty)(a)} = \delta^s_{(a)} \delta^\mu_{(\mu)} b^{(\mu)}_{(\infty)s}(\tau)
$$

with the same asymptotic $b^{(\mu)}_{(\infty)A}(\tau)$ of Section III.

Let us remark that the $\Sigma_\tau$-adapted tetrads in adapted coordinates of Eqs.(40) of I, are ${}^4_{(\Sigma)}\check{\tilde{E}}^A_{(\mu)}$ with

$$
{}^4_{(\Sigma)}\check{\tilde{E}}^\tau_{(\mu)} = \big(\frac{1}{N}; -{}^3e^r_{(a)}\, {}^3e^s_{(a)} \frac{N_s}{N}\big) \qquad , {}^4_{(\Sigma)}\check{\tilde{E}}^r_{(\mu)} = (0; {}^3e^r_{(a)}).
$$

Due to the presence of the lapse function in the denominator which is linearly increasing in $\vec{\sigma}$ [to have the possibility of defining $J^{AB}_{ADM}$], these adapted tetrads exist without singularities

at spatial infinity only if $\tilde{\lambda}_{AB}(\tau) = 0$, i.e. on WSW hypersurfaces. The same happens for the adapted cotetrads ${}^4_{(\Sigma)}\check{\tilde{E}}^{(\mu)}_A$ with ${}^4_{(\Sigma)}\check{\tilde{E}}^{(o)}_A = (N; 0)$, ${}^4_{(\Sigma)}\check{\tilde{E}}^{(a)}_A = (N^{(a)} = N_{(a)}; {}^3e^{(a)}_r = {}^3e_{(a)r})$. Therefore, it seems again that tetrad gravity, without supertranslations and with Poincaré charges, admits well defined adapted tetrads and cotetrads (with components in adapted holonomic coordinates) only after having been restricted to WSW hypersurfaces (rest frame), whose asymptotic normals $l^{(\mu)}_{(\infty)} = l^{(\mu)}_{(\infty)\Sigma}$ , tangent to $S_\infty$, are parallel to $\hat{P}^{(\mu)}_{ADM} = b^{(\mu)}_{(\infty)A}\hat{P}^A_{ADM}$ with $\hat{P}^r_{ADM} \approx 0$ [namely when one is inside the Christodoulou-Klainermann class of solutions]. Let us remember from the end of Section III that this implies the existence of an inertial system at spatial infinity when $\tilde{\lambda}_A(\tau) = (\epsilon; \vec{0})$ and $\tilde{\lambda}_{AB}(\tau) = 0$, namely the absence of accelerations and rotations there [when $\tilde{\lambda}_A(\tau) \neq 0$ there is a direction independent global acceleration of the origin $x^{(\mu)}_{(\infty)}(\tau)$, since $\dot{x}^{(\mu)}_{(\infty)}(\tau) = b^{(\mu)}_{(\infty)A}\tilde{\lambda}_A(\tau)$].

In tetrad gravity we shall assume the following boundary conditions consistent with Eqs.(20) and (24) of metric gravity [$\epsilon > 0$]

$$\begin{aligned}
{}^3e_{(a)r}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} (1+\frac{M}{2r})\delta_{(a)r} + {}^3w_{(a)r}(\tau,\vec{\sigma}), \qquad {}^3w_{(a)r}(\tau,\vec{\sigma}) = o_4(r^{-3/2}),\\
{}^3e^r_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} (1-\frac{M}{2r})\delta^r_{(a)} + {}^3w^r_{(a)}(\tau,\vec{\sigma}), \qquad {}^3w^r_{(a)}(\tau,\vec{\sigma}) = o_4(r^{-3/2}),\\
{}^3g_{rs}(\tau,\vec{\sigma}) &= [{}^3e_{(a)r}\,{}^3e_{(a)s}](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} (1+\frac{M}{r})\delta_{rs} + {}^3h_{rs}(\tau,\vec{\sigma}),\\
{}^3h_{rs}(\tau,\vec{\sigma}) &= \frac{1}{r}[\delta_{(a)r}\,{}^3w_{(as)(a)s}(\tau,\vec{\sigma}) + {}^3w_{(as)(a)r}(\tau,\vec{\sigma})\delta_{(a)s}] + O(r^{-2}) = o(r^{-3/2}),
\end{aligned}$$

$$\begin{aligned}
{}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} o_3(r^{-5/2}),\\
{}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma}) &= \frac{1}{4}[{}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)} + {}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} {}^3\tilde{k}^{rs}(\tau,\vec{\sigma}) = o_3(r^{-5/2}),
\end{aligned}$$

$$\begin{aligned}
N(\tau,\vec{\sigma}) &= N_{(as)}(\tau,\vec{\sigma}) + n(\tau,\vec{\sigma}),\\
n(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(2+\epsilon)}),\\
N_{(as)}(\tau,\vec{\sigma}) &= -\tilde{\lambda}_\tau(\tau) - \frac{1}{2}\tilde{\lambda}_{\tau s}(\tau)\sigma^s,
\end{aligned}$$

$$\begin{aligned}
N_r(\tau,\vec{\sigma}) &= N_{(as)r}(\tau,\vec{\sigma}) + n_r(\tau,\vec{\sigma}),\\
n_r(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
N_{(as)r}(\tau,\vec{\sigma}) &= -\tilde{\lambda}_r(\tau) - \frac{1}{2}\tilde{\lambda}_{rs}(\tau)\sigma^s,\\
N_{(a)}(\tau,\vec{\sigma}) &= {}^3e^r_{(a)}(\tau,\vec{\sigma})N_r(\tau,\vec{\sigma}) = \sum_r \delta^r_{(a)} e^{-q(\tau,\vec{\sigma})-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})} N_r(\tau,\vec{\sigma}) =\\
&= N_{(as)(a)}(\tau,\vec{\sigma}) + n_{(a)}(\tau,\vec{\sigma}),\\
n_{(a)}(\tau,\vec{\sigma}) &= [{}^3e^r_{(a)}n_r](\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),
\end{aligned}$$

$$\begin{aligned}
\tilde{\pi}^n(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-3}),\\
\tilde{\pi}_{\vec{n},(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-3}),
\end{aligned}$$

$$
\begin{aligned}
\lambda_n(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(3+\epsilon)}),\\
\lambda_{\vec{n},(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
\beta(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(3+\epsilon)}),\\
\beta^r(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}),\\
\\
\hat{\mathcal{H}}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-3}),\\
{}^3\tilde{\Theta}_{\check{r}}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-3}),\\
\\
{}^3\tilde{M}_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-6}),\\
\alpha_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(1+\epsilon)}),\\
\hat{\mu}_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(1+\epsilon)}),\\
\\
\varphi_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(1+\epsilon)}),\\
\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-2}),\\
\lambda^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-(1+\epsilon)}).
\end{aligned}
\tag{34}
$$

With these boundary conditions all proper gauge transformations [generated by $\tilde{\mathcal{H}}(\tau,\vec{\sigma})$ with parameter $\beta(\tau,\vec{\sigma}) \rightarrow O(r^{-(3+\epsilon)})$, $\tilde{\Theta}_r(\tau,\vec{\sigma})$ with $\beta^{\check{r}}(\tau,\vec{\sigma}) \rightarrow O(r^{-\epsilon})$, ${}^3\tilde{M}_{(a)}(\tau,\vec{\sigma})$ with $\alpha_{(a)}(\tau,\vec{\sigma}) \rightarrow O(r^{-(1+\epsilon)})$, $\tilde{\pi}^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma})$ with $\varphi_{(a)}(\tau,\vec{\sigma}) \rightarrow O(r^{-(1+\epsilon)})$ for $r \rightarrow \infty$] go asymptotically to the identity.

Near spatial infinity there is a dynamical preferred observer [either the canonical non-covariant Newton-Wigner-like position $\tilde{x}^{(\mu)}_{(\infty)}(\tau)$ or the covariant non-canonical origin of asymptotic Cartesian coordinates $x^{(\mu)}_{(\infty)}(\tau)$] with an associated asymptotic inertial (or Lorentz) reference frame given by the asymptotic limit of the $\Sigma_\tau$-adapted tetrads of Eqs.(40) of I: however, as said, these asymptotic tetrads are well defined only in absence of super-translations on the rest-frame WSW hypersurfaces, where (modulo a rigid 3-rotation) we get

$$
\begin{aligned}
&{}^4_{(\infty\Sigma)}\check{\tilde{E}}^{\tau}_{(\alpha)} = (\tfrac{1}{N_{(as)}(\tau)};\vec{0}),\; {}^4_{(\infty\Sigma)}\check{\tilde{E}}^{r}_{(\alpha)} = (-\tfrac{N^r_{(as)}(\tau)}{N_{(as)}(\tau)};\delta^r_{(a)}),\\
&{}^4_{(\infty\Sigma)}\check{\tilde{E}}^{(\alpha)}_{\tau} = (N_{(as)}(\tau);\delta^{(a)}_r N^r_{(as)}(\tau)),\; {}^4_{(\infty\Sigma)}\check{\tilde{E}}^{(\alpha)}_{r} = (0;\delta^{(a)}_r).
\end{aligned}
$$

The associated asymptotic triads are the possible asymptotic limits of the Frauendiener triads.

Then, following the scenario b) of Section III, the differentiable and finite Dirac Hamiltonian is assumed to be [from paper I and Eqs.(51) and (70) of II we have $N_{\check{r}}\,{}^3\tilde{\mathcal{H}}^{\check{r}} \approx -N_{(a)}\,{}^3e^{\check{s}}_{(a)}\,{}^3\tilde{\Theta}_{\check{s}} \approx -N^{\check{s}}\,{}^3\tilde{\Theta}_{\check{s}} = -N_{(a)}\,{}^3e^{s}_{(a)}\,{}^3\tilde{\Theta}_s \approx -N_{(a)}\,{}^3e^s_{(a)}\frac{\partial\xi^r}{\partial\sigma^s}\tilde{\pi}^{\vec{\xi}}_r = -N_u\,{}^3g^{us}\frac{\partial\xi^r}{\partial\sigma^s}\tilde{\pi}^{\vec{\xi}}_r = -\tilde{N}^r\tilde{\pi}^{\vec{\xi}}_r$]

$$
\begin{aligned}
\hat{H}_{(D)ADM} = \int d^3\sigma [n\hat{\mathcal{H}} - \tilde{n}^r\tilde{\pi}^{\vec{\xi}}_r + \lambda^{\vec{\varphi}}_{(a)}\tilde{\pi}^{\vec{\varphi}}_{(a)} + \hat{\mu}_{(a)}\,{}^3\tilde{M}_{(a)} +\\
+ \lambda_n\tilde{\pi}^n + \lambda^{\vec{n}}_{(a)}\tilde{\pi}^{\vec{n}}_{(a)}](\tau,\vec{\sigma}) +
\end{aligned}
$$

$$+\tilde{\lambda}_A(\tau)[p^A_{(\infty)}-\hat{P}^A_{ADM}]+\frac{1}{2}\tilde{\lambda}_{AB}(\tau)[J^{AB}_{(\infty)}-\hat{J}^{AB}_{ADM}], \tag{35}$$

with the same weak Poincaré charges of metric gravity, Eqs. (14), expressed in terms of cotriads ${}^3e_{(a)r}$ and their conjugate momenta ${}^3\tilde{\pi}^r_{(a)}$, by using ${}^3g_{rs}={}^3e_{(a)r}\,{}^3e_{(a)s}$, ${}^3\tilde{\Pi}^{rs}=\frac{1}{4}[{}^3e^r_{(a)}\,{}^3\tilde{\pi}^s_{(a)}+{}^3e^s_{(a)}\,{}^3\tilde{\pi}^r_{(a)}]$ (see Eq.(84) of I).

Let us remark that, since we are using the ADM expression for the energy $\hat{P}^\tau_{ADM}$, we have not to show that it is definite positive, because the ADM canonical approach to metric gravity is contained in the one to tetrad gravity.

In the 3-orthogonal gauges of Section V of II and in the final canonical basis [$q$, $\rho$, $r_{\bar{a}}$, $\pi_{\bar{a}}$], one has (before the restriction to WSW hypersurfaces):

i) $\alpha_{(a)}(\tau,\vec{\sigma})=\varphi_{(a)}(\tau,\vec{\sigma})=0$ [so that $\lambda^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma})=\hat{\mu}_{(a)}(\tau,\vec{\sigma})=0$ in the Dirac Hamiltonian], i.e. for the sake of simplicity we choose the timelike congruence of observers' worldlines with the normal $l^A(\tau,\vec{\sigma})$ to $\Sigma_\tau$ as 4-velocity field;
ii) $N(\tau,\vec{\sigma})=[N_{(as)}+n](\tau,\vec{\sigma})=-\tilde{\lambda}_\tau(\tau)-\frac{1}{2}\tilde{\lambda}_{\tau s}(\tau)\sigma^s+n(\tau,\vec{\sigma})$ as the total lapse function;
iii) The gauge fixings $\xi^r(\tau,\vec{\sigma})-\sigma^r\approx 0$ of Section V of II (choice of the 3-orthogonal coordinates) now do not imply $n_r(\tau,\vec{\sigma})\approx 0$ as in II, due to the modification introduced by the addition of the surface terms to the Dirac Hamiltonian. Instead they imply the following results for the total shift function [the Poincaré charges of Eq.(14) have to be used in this equations]

$$\begin{aligned}
&\partial_\tau[\xi^r(\tau,\vec{\sigma})-\sigma^r]\,\circeq\,\{\xi^r(\tau,\vec{\sigma}),\hat{H}_{(D)ADM}\}=\Big[n^s\,\frac{\partial\xi^r}{\partial\sigma^s}\Big](\tau,\vec{\sigma})-\\
&\qquad-\tilde{\lambda}_A(\tau)\{\xi^r(\tau,\vec{\sigma}),\hat{P}^A_{ADM}\}-\frac{1}{2}\tilde{\lambda}_{AB}(\tau)\{\xi^r(\tau,\vec{\sigma}),\hat{J}^{AB}_{ADM}\}\approx 0,\\
&\Rightarrow\qquad n_r(\tau,\vec{\sigma})-\hat{n}_r(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A,\tilde{\lambda}_{AB}]\approx 0,\\
&\qquad\hat{n}_r(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A,\tilde{\lambda}_{AB}]={}^3g_{rs}(\tau,\vec{\sigma})\Big[\tilde{\lambda}_A(\tau)\{\xi^s(\tau,\vec{\sigma}),\hat{P}^A_{ADM}\}+\\
&\qquad+\frac{1}{2}\tilde{\lambda}_{AB}(\tau)\{\xi^s(\tau,\vec{\sigma}),\hat{J}^{AB}_{ADM}\}\Big],\\
&\qquad\partial_\tau\Big[n_r(\tau,\vec{\sigma})-\hat{n}_r(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A,\tilde{\lambda}_{AB}]\Big]\approx\\
&\qquad\approx[\lambda^{\vec{n}}_{(a)}\,{}^3\hat{e}_{(a)r}](\tau,\vec{\sigma})-\\
&\qquad-\{\hat{n}_r(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A,\tilde{\lambda}_{AB}],\hat{H}_{(D)ADM}\}\approx 0,\\
&\Rightarrow\qquad\lambda^{\vec{n}}_{(a)}(\tau,\vec{\sigma})\quad\textit{determined}.
\end{aligned}\tag{36}$$

Therefore, the shift functions do not vanish in the 3-orthogonal gauges avoiding the "synchronous" coordinates with their tendency to develop coordinate singularities in short times.
iv) After going to Dirac brackets with respect to all the second class constraints implied by

the previous gauge fixings (for the sake of simplicity they will always be denoted $\{.,.\}]$ we remain with the Dirac Hamiltonian

$$\hat{H}_{(D)ADM,R} = \int d^3\sigma [n\hat{\mathcal{H}}_R + \lambda_n \tilde{\pi}^n](\tau,\vec{\sigma}) + \tilde{\lambda}_A(\tau)[p^A_{(\infty)} - \hat{P}^A_{ADM}] + \frac{1}{2}\tilde{\lambda}_{AB}(\tau)[J^{AB}_{(\infty)} - \hat{J}^{AB}_{ADM}].$$

As shown in II, the surviving canonical variables in 3-orthogonal gauges are $n$, $\tilde{\pi}^n$, $q$, $\rho$, $r_{\bar{a}}$, $\pi_{\bar{a}}$.

From Eqs.(99), (102), (84), (90), (95), (96) of II [$\gamma_{PP_1}$ is the geodesic between P and $P_1$ for the 3-metric; $\phi = e^{q/2}$] we get

$$\begin{aligned}
{}^3\hat{e}_{(a)r}(\tau,\vec{\sigma}) &= \delta_{(a)r}(e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}})(\tau,\vec{\sigma}),\\
{}^3\hat{g}_{rs}(\tau,\vec{\sigma}) &= [e^{2q+\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}](\tau,\vec{\sigma})\delta_{rs},\\
{}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) &= \sum_s \int d^3\sigma_1 \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|q,r_{\bar{a}}](e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}})(\tau,\vec{\sigma}_1)\\
&\quad \Big[\frac{1}{3}\rho + \sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}s}\pi_{\bar{b}}\Big](\tau,\vec{\sigma}_1)\\
\rightarrow_{\rho\rightarrow 0} &\quad \sqrt{3}\sum_s\sum_{\bar{b}}\gamma_{\bar{b}s}\int d^3\sigma_1 \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,r_{\bar{a}}](\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\pi_{\bar{b}})(\tau,\vec{\sigma}_1),\\
\mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}] &= \delta^r_{(a)}\delta^r_s\delta^3(\vec{\sigma},\vec{\sigma}_1) + \mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}],\\
\mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|q,r_{\bar{a}}] &= \frac{1}{2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}r}r_{\bar{c}}(\tau,\vec{\sigma})}\Big[\sum_{w\neq s}\delta_{(k)w}e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}(\gamma_{\bar{c}w}-\gamma_{\bar{c}s})r_{\bar{c}}(\tau,\vec{\sigma}_1)}\,.\\
&\quad \cdot\Big(\frac{\partial q(\tau,\vec{\sigma}_1)}{\partial\sigma_1^w} + \frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}s}\frac{\partial r_{\bar{c}}(\tau,\vec{\sigma}_1)}{\partial\sigma_1^w}\Big)e^{-q(\tau,\vec{\sigma})}\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) +\\
&\quad +\delta_{(k)s}\frac{\partial}{\partial\sigma_1^s}e^{-q(\tau,\vec{\sigma})}\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau)\Big],\\
e^{-q(\tau,\vec{\sigma})} &\quad \delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) =\\
&= e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}r}r_{\bar{c}}(\tau,\vec{\sigma})}d^r_{\gamma_{PP_1}}\Big(P_{\gamma_{PP_1}}e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}}d\sigma_2^w\,{}^3\hat{\omega}_{w(c)}(\tau,\vec{\sigma}_2)\hat{R}^{(c)}}\Big)_{(a)(k)} +\\
&+ \sum_u \delta_{(a)u}e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u}r_{\bar{c}}(\tau,\vec{\sigma})}d^u_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)\\
&\quad \delta^r_{(b)}\Big(P_{\gamma_{PP_1}}e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}}d\sigma_2^w\,{}^3\hat{\omega}_{w(c)}(\tau,\vec{\sigma}_2)\hat{R}^{(c)}}\Big)_{(b)(k)},\\
{}^3\hat{\omega}_{t(d)}(\tau,\vec{\sigma}) &= \epsilon_{(d)(m)(n)}\delta_{(m)t}\delta_{(n)u}(e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}t}-\gamma_{\bar{a}u})r_{\bar{a}}}[\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}t}\partial_u r_{\bar{b}}])(\tau,\vec{\sigma}),\\
&\quad \Downarrow\\
q(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} \frac{M}{2r} + o_4(r^{-3/2}),\\
\phi(\tau,\vec{\sigma}) &= e^{q(\tau,\vec{\sigma})/2} \rightarrow_{r\rightarrow\infty} 1 + \frac{M}{4r} + o_4(r^{-3/2}),\\
r_{\bar{a}}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} o_4(r^{-3/2}),
\end{aligned}$$

$$
\begin{aligned}
\rho(\tau,\vec{\sigma}) &= \frac{1}{2}\phi(\tau,\vec{\sigma})\pi_\phi(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} o_3(r^{-5/2}),\\
\pi_{\bar{a}}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} o_3(r^{-3}),\\
{}^3\hat{\omega}_{r(a)}(\tau,\vec{\sigma}) &\rightarrow_{r\rightarrow\infty} O(r^{-2}),
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
{}^3\hat{K}_{rs}(\tau,\vec{\sigma}) &= \frac{\epsilon}{4k}[e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}(\gamma_{\bar{c}r}+\gamma_{\bar{c}s})r_{\bar{c}}}\sum_u(\delta_{ru}\delta_{(a)s}+\delta_{su}\delta_{(a)r}-\delta_{rs}\delta_{(a)u})\\
&\quad e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u}r_{\bar{c}}}\,{}^3\hat{\tilde{\pi}}^u_{(a)}](\tau,\vec{\sigma}),\\
{}^3\hat{K}(\tau,\vec{\sigma}) &= -\frac{\epsilon}{4k}[e^{-2q}\sum_u\delta_{(a)u}e^{\frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u}r_{\bar{c}}}\,{}^3\hat{\tilde{\pi}}^u_{(a)}](\tau,\vec{\sigma}) =\\
&= -\frac{\epsilon}{4k}e^{-3q(\tau,\vec{\sigma})}\{\rho(\tau,\vec{\sigma})+\sum_u(e^{q+\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}})(\tau,\vec{\sigma})\int d^3\sigma_1\delta_{(a)u}\\
&\quad \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|q,r_{\bar{a}}](e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}s}r_{\bar{a}}})(\tau,\vec{\sigma}_1)[\frac{1}{3}\rho+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}](\tau,\vec{\sigma}_1)\},\\
{}^3\hat{\tilde{\Pi}}^{rs}(\tau,\vec{\sigma}) &= \frac{1}{4}[{}^3\hat{e}^r_{(a)}\,{}^3\hat{\tilde{\pi}}^s_{(a)}+{}^3\hat{e}^s_{(a)}\,{}^3\hat{\tilde{\pi}}^r_{(a)}](\tau,\vec{\sigma}) =\\
&= \frac{1}{4}e^{-q(\tau,\vec{\sigma})}[e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\delta^r_{(a)}\,{}^3\hat{\tilde{\pi}}^s_{(a)}+e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\delta^s_{(a)}\,{}^3\hat{\tilde{\pi}}^r_{(a)}](\tau,\vec{\sigma}),\\
{}^3\hat{\Gamma}^r_{uv}(\tau,\vec{\sigma}) &= \Big(-\delta_{uv}\sum_s\delta^r_s e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}-\gamma_{\bar{a}s})r_{\bar{a}}}[\partial_s q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_s r_{\bar{b}}]+\\
&\quad +\delta^r_u[\partial_v q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_v r_{\bar{b}}]+\delta^r_v[\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}v}\partial_u r_{\bar{b}}]\Big)(\tau,\vec{\sigma}),\\
{}^3\hat{\Gamma}^u_{uv}(\tau,\vec{\sigma}) &= 3\partial_v q(\tau,\vec{\sigma}),\\
{}^3\hat{G}_{rsuv}(\tau,\vec{\sigma}) &= [{}^3\hat{g}_{ru}\,{}^3\hat{g}_{sv}+{}^3\hat{g}_{rv}\,{}^3\hat{g}_{su}-{}^3\hat{g}_{rs}\,{}^3\hat{g}_{uv}](\tau,\vec{\sigma}) =\\
&= e^{4q(\tau,\vec{\sigma})}[e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}s})r_{\bar{a}}}(\delta_{ru}\delta_{sv}+\delta_{rv}\delta_{su})-e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}+\gamma_{\bar{a}u})r_{\bar{a}}}\delta_{rs}\delta_{uv}](\tau,\vec{\sigma}),
\end{aligned}
\tag{38}
$$

where the second equation can be read as an integral equation to get $\rho(\tau,\vec{\sigma})$, the momentum conjugate to the conformal factor, in terms of ${}^3\hat{K}(\tau,\vec{\sigma})$, $q(\tau,\vec{\sigma})=\frac{1}{6}ln\,{}^3\hat{g}(\tau,\vec{\sigma})$ and $r_{\bar{a}}(\tau,\vec{\sigma})=\frac{\sqrt{3}}{2}\sum_r\gamma_{\bar{a}r}ln\,[{}^3\hat{g}_{rr}/{}^3\hat{g}](\tau,\vec{\sigma})$ [see Eqs.(101) of II], in the 3-orthogonal gauges.

The expression of ${}^3\hat{K}_{rs}$ in these gauges replaces the knowledge of the gravitomagnetic potential $\vec{W}$ in the York TT decomposition of the extrinsic curvature in the conformal approach (see Appendix C of II; the three degrees of freedom of $\vec{W}$ correspond to the three eliminated parameters $\vec{\xi}$ of pseudo-diffeomorphisms).

We can now write explicitly the equations determining $n_r(\tau,\vec{\sigma})$ for $\tilde{\lambda}_{AB}(\tau)=0$, i.e. on the WSW hypersurfaces. From Eqs.(79) or (90) of II and from ${}^3\tilde{\Pi}^{uv}=\frac{1}{4}[{}^3e^u_{(a)}\,{}^3\tilde{\pi}^v_{(a)}+{}^3e^v_{(a)}\,{}^3\tilde{\pi}^u_{(a)}]$ (see I) we get

$$
\begin{aligned}
&\{\xi^r(\tau,\vec{\sigma}),{}^3\tilde{\pi}^s_{(a)}(\tau,\vec{\sigma}_1)\}|_{\vec{\xi}=\vec{\sigma}}\\
&= -\tfrac{1}{2}\,{}^3\hat{e}^s_{(b)}(\tau,\vec{\sigma}_1)\Big[{}^3\hat{e}_{(b)w}(\tau,\vec{\sigma}_1)\zeta^{(\hat{\omega})w}_{(a)(c)}(\vec{\sigma}_1,\vec{\sigma},\tau)+{}^3\hat{e}_{(a)w}(\tau,\vec{\sigma}_1)\zeta^{(\hat{\omega})w}_{(b)(c)}(\vec{\sigma}_1,\vec{\sigma},\tau)\Big]{}^3\hat{e}^r_{(c)}(\tau,\vec{\sigma})
\end{aligned}
$$

$$\{\xi^r(\tau,\vec\sigma), {}^3\tilde\Pi^{uv}(\tau,\vec\sigma_1)\} = -\frac{1}{4}[{}^3\hat e^u_{(a)}\delta^v_w + {}^3\hat e^v_{(a)}\delta^u_w](\tau,\vec\sigma_1)\zeta^{(\hat\omega)w}_{(a)(b)}(\vec\sigma_1,\vec\sigma,\tau)\,{}^3\hat e^r_{(b)}(\tau,\vec\sigma).$$

By using Eqs.(14) we get

$$\{\xi^r(\tau,\vec\sigma), \hat P^\tau_{ADM}\} = -\frac{1}{4k}\,{}^3G_{o(a)(b)(c)(d)}\int d^3\sigma_1 \frac{{}^3\hat e_{(d)s}\,{}^3\hat e_{(b)v}\,{}^3\hat{\tilde\pi}^s_{(c)}}{\sqrt{\gamma}}(\tau,\vec\sigma_1)\zeta^{(\hat\omega)v}_{(a)(e)}(\vec\sigma_1,\vec\sigma,\tau)\,{}^3\hat e^r_{(e)}(\tau,\vec\sigma)$$

$$\{\xi^r(\tau,\vec\sigma), \hat P^s_{ADM}\} = \int d^3\sigma_1\,{}^3\hat\Gamma^s_{uv}(\tau,\vec\sigma_1)\,{}^3\hat e^u_{(a)}(\tau,\vec\sigma_1)\,\zeta^{(\hat\omega)v}_{(a)(b)}(\vec\sigma_1,\vec\sigma,\tau)\,{}^3\hat e^r_{(b)}(\tau,\vec\sigma).$$

Therefore, by using Eqs.(38) we get $[\phi = e^{q/2}]$

$$\begin{aligned}
n_r(\tau,\vec\sigma) &\approx \hat n_r(\tau,\vec\sigma|r_{\bar a},\pi_{\bar a},\tilde\lambda_A,0] =\\
&= \Big[\phi^2 e^{\frac{1}{\sqrt3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\Big](\tau,\vec\sigma)\Big(-\frac{\tilde\lambda_\tau(\tau)}{4k}\,{}^3G_{o(a)(b)(c)(d)}\delta_{(b)v}\delta_{(d)r}\\
&\int d^3\sigma_1\Big[\phi^{-6}e^{\frac{1}{\sqrt3}\sum_{\bar a}(\gamma_{\bar ar}-\gamma_{\bar av})r_{\bar a}}\,{}^3\hat{\tilde\pi}^r_{(c)}\Big](\tau,\vec\sigma_1)\zeta^{(\hat\omega)v}_{(a)(e)}(\vec\sigma_1.\vec\sigma,\tau)\delta^r_{(e)} +\\
&+\tilde\lambda_r(\tau)\int d^3\sigma_1\delta^u_{(a)}\,{}^3\hat\Gamma^r_{uv}(\tau,\vec\sigma_1)\zeta^{(\hat\omega)v}_{(a)(b)}(\vec\sigma_1,\vec\sigma,\tau)\delta^r_{(b)}\Big),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\qquad N_{(a)}(\tau,\vec\sigma)|_{\tilde\lambda_{AB}=0} &= [{}^3\hat e^r_{(a)}N_r](\tau,\vec\sigma)|_{\tilde\lambda_{AB}=0}\approx\\
&\approx \Big[{}^3\hat e^r_{(a)}\Big(-\tilde\lambda_r(\tau)+\hat n_r[r_{\bar a},\pi_{\bar a},\tilde\lambda_A,\tilde\lambda_{AB}]\Big)\Big](\tau,\vec\sigma).
\end{aligned}\tag{39}$$

The reduced Dirac Hamiltonian and the weak and strong Poincaré charges of Eqs.(13), (14) and (12) become [Eqs.(102) and (104) of II are used]

$$\begin{aligned}
\hat H_{(D)ADM,R} &= \int d^3\sigma[n\hat{\mathcal H}_R + \lambda_n\tilde\pi^n](\tau,\vec\sigma) +\\
&+\tilde\lambda_A(\tau)[p^A_{(\infty)} - \hat P^A_{ADM,R}] + \frac{1}{2}\tilde\lambda_{AB}(\tau)[J^{AB}_{(\infty)} - \hat J^{AB}_{ADM,R}],
\end{aligned}$$

$$\begin{aligned}
\hat P^\tau_{ADM,R} &= \epsilon\int d^3\sigma\Big(k\Big[e^q\sum_r e^{-\frac{1}{\sqrt3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}\times\\
&\Big(\frac{1}{3}\sum_{\bar b\bar c}(2\gamma_{\bar br}\gamma_{\bar cr}+\delta_{\bar br\bar cr})\partial_r r_{\bar b}\partial_r r_{\bar c} - \frac{2}{\sqrt3}(\sum_{\bar b}\gamma_{\bar br}\partial_r r_{\bar b})\partial_r q - (\partial_r q)^2 -\\
&-\sum_u e^{\frac{2}{\sqrt3}\sum_{\bar a}(\gamma_{\bar ar}-\gamma_{\bar au})r_{\bar a}}[\frac{2}{3}\sum_{\bar b\bar c}\gamma_{\bar br}\gamma_{\bar cr}\partial_u r_{\bar b}\partial_u r_{\bar c} +\\
&+\sqrt3(\sum_{\bar b}\gamma_{\bar br}\partial_u r_{\bar b})\partial_u q + (\partial_u q)^2]\Big)\Big](\tau,\vec\sigma) -\\
&-\frac{e^{-q(\tau,\vec\sigma)}}{8k}\Big[(e^{-2q}[6\sum_{\bar a}\pi^2_{\bar a} - \frac{1}{3}\rho^2])(\tau,\vec\sigma) +\\
&+2(e^{-q}\sum_u e^{\frac{1}{\sqrt3}\sum_{\bar a}\gamma_{\bar au}r_{\bar a}}[2\sqrt3\sum_{\bar b}\gamma_{\bar bu}\pi_{\bar b} - \frac{1}{3}\rho])(\tau,\vec\sigma)\times\\
&\int d^3\sigma_1\sum_r \delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt3}\sum_{\bar a}\gamma_{\bar ar}r_{\bar a}}[\frac{\rho}{3}+\sqrt3\sum_{\bar b}\gamma_{\bar br}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1) +\\
&+\int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt3}\sum_{\bar a}\gamma_{\bar au}+r_{\bar a}(\tau,\vec\sigma)}\times
\end{aligned}$$

$$
\begin{aligned}
&\sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}]\Big)(\tau,\vec{\sigma}_1)\times\\
&\sum_s \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_2,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}]\Big)(\tau,\vec{\sigma}_2)+\\
&+\sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}+\gamma_{\bar{a}v})r_{\bar{a}}(\tau,\vec{\sigma})}(\delta^u_{(b)}\delta^v_{(a)}-\delta^u_{(a)}\delta^v_{(b)})\times\\
&\sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}]\Big)(\tau,\vec{\sigma}_1)\\
&\sum_s \mathcal{T}^v_{(b)s}(\vec{\sigma},\vec{\sigma}_2,\tau|q,r_{\bar{a}}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}]\Big)(\tau,\vec{\sigma}_2)\Big)\Big]\Big),\\
\hat{P}^r_{ADM,R} &= \int d^3\sigma e^{q(\tau,\vec{\sigma})}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})}\Big(e^{-q(\tau,\vec{\sigma})}\\
&\Big[\sum_u e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_r r_{\bar{b}})(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}u}\pi_{\bar{c}})-\\
&-2e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_r r_{\bar{b}})(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}r}\pi_{\bar{c}})\Big](\tau,\vec{\sigma})+\\
&+\sum_{uv}\int d^3\sigma_1\Big[(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_r r_{\bar{b}}])(\tau,\vec{\sigma})\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]-\\
&-(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_u r_{\bar{b}})(\tau,\vec{\sigma})\Big(\delta_{r(a)}\mathcal{T}^u_{(a)v}+\delta_{u(a)}\mathcal{T}^r_{(a)v}\Big)(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big]\\
&\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}v}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}v}\pi_{\bar{c}}]\Big)(\tau,\vec{\sigma}_1)\Big),\\
\hat{J}^{rs}_{ADM,R} &= \frac{1}{2}\int d^3\sigma e^{-2q(\tau,\vec{\sigma})}\Big[\sigma^s e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})}\\
&\Big[\sum_u e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_r r_{\bar{b}})(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}u}\pi_{\bar{c}})-\\
&-2e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_r r_{\bar{b}})(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}r}\pi_{\bar{c}})\Big](\tau,\vec{\sigma})-\\
&-\sigma^r e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}(\tau,\vec{\sigma})}\\
&\Big[\sum_u e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}(\partial_s q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_s r_{\bar{b}})(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}u}\pi_{\bar{c}})-\\
&-2e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}(\partial_s q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}s}\partial_s r_{\bar{b}})(\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}})\Big](\tau,\vec{\sigma})\Big]+\\
&+\frac{1}{2}\sum_{uv}\int d^3\sigma d^3\sigma_1 e^{-q(\tau,\vec{\sigma})}\Big[\sigma^s e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})}\\
&\Big[(\partial_r q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_r r_{\bar{b}}](\tau,\vec{\sigma})\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]]-\\
&-(\partial_u q+\frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_u r_{\bar{b}})(\tau,\vec{\sigma})\Big(\delta_{r(a)}\mathcal{T}^u_{(a)v}+\delta_{u(a)}\mathcal{T}^r_{(a)v}\Big)(\vec{\sigma},\vec{\sigma}_1,\tau|q,r_{\bar{a}}]\Big]-\\
&-\sigma^r e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}(\tau,\vec{\sigma})}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[(\partial_s q + \frac{1}{\sqrt{3}}\sum_{\bar b} \gamma_{\bar b u}\partial_s r_{\bar b}](\tau,\vec\sigma)\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]] -\\
&- (\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_u r_{\bar b})(\tau,\vec\sigma)\Big(\delta_{s(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^s_{(a)v}\Big)(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big]\Big]\\
&(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c v}\pi_{\bar c}])(\tau,\vec\sigma_1),
\end{aligned}
$$

$$
\begin{aligned}
\hat J^{\tau r}_{ADM,R} &= \epsilon\int d^3\sigma\,\sigma^r\Big(k\Big[e^q\sum_r e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}\times\\
&\Big(\frac{1}{3}\sum_{\bar b\bar c}(2\gamma_{\bar b r}\gamma_{\bar c r} + \delta_{\bar b r\bar c r})\partial_r r_{\bar b}\partial_r r_{\bar c} - \frac{2}{\sqrt{3}}(\sum_{\bar b}\gamma_{\bar b r}\partial_r r_{\bar b})\partial_r q - (\partial_r q)^2 -\\
&- \sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r}-\gamma_{\bar a u})r_{\bar a}}[\frac{2}{3}\sum_{\bar b\bar c}\gamma_{\bar b r}\gamma_{\bar c r}\partial_u r_{\bar b}\partial_u r_{\bar c} +\\
&+ \sqrt{3}(\sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b})\partial_u q + (\partial_u q)^2]\Big)\Big](\tau,\vec\sigma) -\\
&- \frac{e^{-q(\tau,\vec\sigma)}}{8k}\Big[(e^{-2q}[6\sum_{\bar a}\pi^2_{\bar a} - \frac{1}{3}\rho^2])(\tau,\vec\sigma) +\\
&+ 2(e^{-q}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}r_{\bar a}}[2\sqrt{3}\sum_{\bar b}\gamma_{\bar b u}\pi_{\bar b} - \frac{1}{3}\rho])(\tau,\vec\sigma)\times\\
&\int d^3\sigma_1\sum_r\delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1) +\\
&+ \int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}+r_{\bar a}(\tau,\vec\sigma)}\times\\
&\sum_r\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1)\times\\
&\sum_s\mathcal{T}^u_{(a)s}(\vec\sigma,\vec\sigma_2,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c s}\pi_{\bar c}]\Big)(\tau,\vec\sigma_2) +\\
&+ \sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a u}+\gamma_{\bar a v})r_{\bar a}(\tau,\vec\sigma)}(\delta^u_{(b)}\delta^v_{(a)} - \delta^u_{(a)}\delta^v_{(b)})\times\\
&\sum_r\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1)\\
&\sum_s\mathcal{T}^v_{(b)s}(\vec\sigma,\vec\sigma_2,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c s}\pi_{\bar c}]\Big)(\tau,\vec\sigma_2)\,\Big)\,\Big]\,\Big) +\\
&+ \epsilon k\int d^3\sigma\Big(e^{-q-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}\Big)(\tau,\vec\sigma)\\
&\Big[\sum_v e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v}r_{\bar a}}\Big(e^{2q+\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v}r_{\bar a}} - 1\Big)(\partial_r q + \frac{2}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b r} + \gamma_{\bar b v})\partial_r r_{\bar b}) -\\
&- e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}\Big(e^{2q+\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}} - 1\Big)(\partial_r q + \frac{4}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_r r_{\bar b})\Big](\tau,\vec\sigma),
\end{aligned}
$$

$$
P^\tau_{ADM,R} = \hat P^\tau_{ADM,R} + \int d^3\sigma\hat{\mathcal{H}}_R(\tau,\vec\sigma) =
$$

$$
\begin{aligned}
&= 2\epsilon k \sum_u \int_{S^2_{\tau,\infty}} d^2\Sigma_u \Big( e^{q-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}} [-2\partial_u q + \frac{1}{\sqrt{3}} \sum_{\bar b} \gamma_{\bar b u}\partial_u r_{\bar b}]\Big)(\tau,\vec\sigma),\\
P^r_{ADM,R} &= \hat P^r_{ADM,R} + \int d^3\sigma \hat{\mathcal{H}}^r(\tau,\vec\sigma) \equiv \hat P^r_{ADM,R} =\\
&= -\int_{S^2_{\tau,\infty}} d^2\Sigma_r \Big[ e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} [\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c r}\pi_{\bar c}]\Big](\tau,\vec\sigma) -\\
&- \frac{1}{2}\sum_u \int_{S^2_{\tau,\infty}} d^2\Sigma_u e^{-q(\tau,\vec\sigma)} \sum_v \int d^3\sigma_1 \Big( e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}(\tau,\vec\sigma)} \delta_{r(a)} \mathcal{T}^u_{(a)v} +\\
&+ e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}(\tau,\vec\sigma)} \delta_{u(a)} \mathcal{T}^r_{(a)v}\Big)(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\\
&\quad (e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v} r_{\bar a}} [\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c v}\pi_{\bar c}])(\tau,\vec\sigma_1)\},\\
J^{rs}_{ADM,R} &= \hat J^{rs}_{ADM,R} + \frac{1}{4}\int d^3\sigma [\sigma^s \hat{\mathcal{H}}^r - \sigma^r \hat{\mathcal{H}}^s](\tau,\vec\sigma) =\\
&= -\frac{1}{2}\sum_u \int_{S^2_{\tau,\infty}} d^2\Sigma_u e^{-q(\tau,\vec\sigma)}\\
&\quad \Big( \delta^r_u \sigma^s [e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} (\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b})] -\\
&- \delta^s_u \sigma^r [e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s} r_{\bar a}} (\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b s}\pi_{\bar b})]\Big)(\tau,\vec\sigma) -\\
&- \frac{1}{4}\sum_u \int_{S^2_{\tau,\infty}} d^2\Sigma_u e^{-q(\tau,\vec\sigma)} \sum_v \int d^3\sigma_1\\
&\quad \Big[ \sigma^s \Big( e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}(\tau,\vec\sigma)} \delta_{r(a)} \mathcal{T}^u_{(a)v} + e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}(\tau,\vec\sigma)} \delta_{u(a)} \mathcal{T}^r_{(a)v}\Big) -\\
&- \sigma^r \Big( e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s} r_{\bar a}(\tau,\vec\sigma)} \delta_{s(a)} \mathcal{T}^u_{(a)v} + e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}(\tau,\vec\sigma)} \delta_{u(a)} \mathcal{T}^s_{(a)v}\Big)\Big]\\
&\quad (\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}](e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v} r_{\bar a}} [\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c v}\pi_{\bar c}](\tau,\vec\sigma_1)\},\\
J^{\tau r}_{ADM,R} &= \hat J^{\tau r}_{ADM,R} + \frac{1}{2}\int d^3\sigma \sigma^r \hat{\mathcal{H}}_R(\tau,\vec\sigma) =\\
&= 2\epsilon k \sum_u \int_{S^2_{\tau,\infty}} d^2\Sigma_u \sigma^r \Big( e^{q-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}} [-2\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_u r_{\bar b}]\Big)(\tau,\vec\sigma) +\\
&+ \epsilon k \int_{S^2_{\tau,\infty}} d^2\Sigma_r \Big( e^{-q-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} \Big[ \sum_n e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a n} r_{\bar a}} (e^{2q+\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a n} r_{\bar a}} - 1) -\\
&- e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} (e^{2q+\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} - 1)\Big]\Big)(\tau,\vec\sigma).
\end{aligned}
\tag{40}
$$

The reduced superhamiltonian constraint [see Eq.(104) of II] becomes

$$
\begin{aligned}
\hat{\mathcal{H}}_R(\tau,\vec\sigma) &= -\epsilon \frac{e^{-q(\tau,\vec\sigma)}}{8k} \Big[ (e^{-2q}[6\sum_{\bar a}\pi^2_{\bar a} - \frac{1}{3}\rho^2])(\tau,\vec\sigma) +\\
&+ 2(e^{-q}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}} [2\sqrt{3}\sum_{\bar b}\gamma_{\bar b u}\pi_{\bar b} - \frac{1}{3}\rho])(\tau,\vec\sigma) \times\\
&\quad \int d^3\sigma_1 \sum_r \delta^u_{(a)} \mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}] \Big( e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} [\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1) +
\end{aligned}
$$

$$
\begin{aligned}
& + \int d^3\sigma_1 d^3\sigma_2 \Big( \sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} + r_{\bar a}(\tau,\vec\sigma)} \times \\
& \quad \sum_r \mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1) \times \\
& \quad \sum_s \mathcal{T}^u_{(a)s}(\vec\sigma,\vec\sigma_2,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}r_{\bar a}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar c}\gamma_{\bar c s}\pi_{\bar c}]\Big)(\tau,\vec\sigma_2) + \\
& + \sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a u}+\gamma_{\bar a v})r_{\bar a}(\tau,\vec\sigma)}(\delta^u_{(b)}\delta^v_{(a)} - \delta^u_{(a)}\delta^v_{(b)}) \times \\
& \quad \sum_r \mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1) \\
& \quad \sum_s \mathcal{T}^v_{(b)s}(\vec\sigma,\vec\sigma_2,\tau|q,r_{\bar a}]\Big(e^{-q-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}r_{\bar a}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar c}\gamma_{\bar c s}\pi_{\bar c}]\Big)(\tau,\vec\sigma_2)\Big)\Big] + \\
& + \epsilon k \sum_{r,s} e^{q(\tau,\vec\sigma)-\frac{1}{\sqrt{3}}\sum_{\bar c}(\gamma_{\bar c r}+\gamma_{\bar c s})r_{\bar c}(\tau,\vec\sigma)}\epsilon_{(a)(b)(c)}\delta_{(a)r}\delta_{(b)s}\,{}^3\hat\Omega_{rs(c)}[q,r_{\bar c}](\tau,\vec\sigma) \approx 0, \\
& with \quad \epsilon k \sum_{r,s} e^{q(\tau,\vec\sigma)-\frac{1}{\sqrt{3}}\sum_{\bar c}(\gamma_{\bar c r}+\gamma_{\bar c s})r_{\bar c}(\tau,\vec\sigma)}\epsilon_{(a)(b)(c)}\delta_{(a)r}\delta_{(b)s}\,{}^3\hat\Omega_{rs(c)}[q,r_{\bar c}](\tau,\vec\sigma) = \\
& = \epsilon k (e^{3q}\,{}^3\hat R[q,r_{\bar a}])(\tau,\vec\sigma). \qquad (41)
\end{aligned}
$$

As already anticipated in II, the gauge variable in which the superhamiltonian constraint (or Lichnerowicz equation) has to be solved, is the conformal factor $q(\tau,\vec\sigma)$ [or better $\phi(\tau,\vec\sigma) = e^{q(\tau,\vec\sigma)/2}$], since, as shown in Appendix A, the surface integral giving the ADM energy depends only on it and not on its conjugate momentum $\rho(\tau,\vec\sigma)$. In every Gauss law, the piece of the secondary first class constraint corresponding to a divergence and giving the "strong" form of the conserved charge as the flux through the surface at infinity of a corresponding density depends on the variable which has to be eliminated in the canonical reduction by using the constraint (the conjugate variable is the gauge variable): once the constraint is solved in this variable, it can be put inside the volume expression of the "weak" form of the conserved charge to obtain its expression in the reduced phase space; the strong ADM energy is the only known charge, associated with a constraint bilinear in the momenta, depending only on the coordinates and not on the momenta, so that $\phi$ and not $\rho$ is the unknown in the Lichnerowicz equation.

It is the gauge-fixing constraint to the superhamiltonian one which fixes the momentum $\rho(\tau,\vec\sigma)$, the last gauge variable. Starting from Lichnerowicz [100], usually one chooses the maximal slicing condition ${}^3K(\tau,\vec\sigma)\approx 0$ for this gauge-fixing; Lichnerowicz has shown that with it the superhamiltonian and supermomentum constraints of metric gravity form a system of 5 elliptic differential equations which can be shown to have one and only one solution; moreover, with this condition Schoen and Yau [54] have shown that the ADM 4-momentum is timelike (i.e. the ADM energy is positive or zero for Minkowski spacetime).

However Schoen-Yau have relaxed the maximal slicing condition in their last proof of the positivity of the ADM energy. Therefore, if there is no contradiction with the existence and unicity of the solution of the reduced Lichnerowicz equation (41) (which is now an integro-differential equation for $\phi$), one can replace ${}^3K(\tau,\vec\sigma)\approx 0$ with the gauge-fixing $\rho(\tau,\vec\sigma)\approx 0$, which is natural in our approach with 3-orthogonal coordinates, saving the positivity of the energy.

This entails that at the level of the Dirac brackets associated with the second class constraints $\tilde{\mathcal{H}}_R(\tau,\vec{\sigma}) \approx 0$, $\rho(\tau,\vec{\sigma}) \approx 0$, the physical variables $r_{\bar{a}}(\tau,\vec{\sigma})$, $\pi_{\bar{a}}(\tau,\vec{\sigma})$, remain canonical and describe the canonical physical degrees of freedom of the gravitational field in this gauge. However, since a closed form of the conformal factor in terms of $r_{\bar{a}}$, $\pi_{\bar{a}}$ as a solution of the superhamiltonian constraint [after having put $\rho(\tau,\vec{\sigma}) = 0$ in it] is not known, the ADM energy (weakly coinciding with the ADM invariant mass in the rest-frame instant form) cannot be explicitly expressed in terms of the physical degrees of freedom of the gravitational field in 3-orthogonal coordinates.

It seems difficult to be able to implement the last step of the programme, namely to find the final Shanmugadhasan canonical transformation $q$, $\rho$, $r_{\bar{a}}$, $\pi_{\bar{a}} \mapsto \hat{\mathcal{H}}^{'}_R$, $\rho^{'}$, $r^{'}_{\bar{a}}$, $\pi^{'}_{\bar{a}}$ [$\hat{\mathcal{H}}^{'}_R(\tau,\vec{\sigma}) \approx 0$ equivalent to $\hat{\mathcal{H}}_R(\tau,\vec{\sigma}) \approx 0$ but with $\{\hat{\mathcal{H}}^{'}_R(\tau,\vec{\sigma}), \hat{\mathcal{H}}^{'}_R(\tau,\vec{\sigma}^{'})\} = 0$], so that all the first class constraints of tetrad gravity appear in the final canonical basis in Abelianized form. Equally difficult is to find the analogue of the York map [101] (see also Appendix C of II) in the 3-orthogonal gauge: $q$, $\rho$, $r_{\bar{a}}$, $\pi_{\bar{a}} \mapsto {}^3\hat{K}^{'}$, $\rho^{(K)}$, $r^{(K)}_{\bar{a}}$, $\pi^{(K)}_{\bar{a}}$ [with ${}^3\hat{K}^{'}$ Abelianized version of ${}^3K$].

To transform the superhamiltonian constraint in the reduced Lichnerowicz equation for the conformal factor, let us use $\phi = e^{q/2}$ [$q, \rho \mapsto \phi = e^{q/2}, \pi_\phi = 2\phi^{-1}\rho$]. We get

$$\begin{aligned}
{}^3\hat{g}_{rs} &= e^{2q + \frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\delta_{rs} \equiv e^{2q}\,{}^3\tilde{g}_{rs} = \phi^4\,{}^3\tilde{g}_{rs}, \qquad {}^3\tilde{g}_{rs} = e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}},\\
&\Rightarrow \sum_r {}^3\tilde{\Gamma}^r_{rs} = 0,\\
{}^3\hat{R} &= e^{-2q}[-4\,{}^3\tilde{g}^{rs}\partial_r\partial_s q - 2\,{}^3\tilde{g}^{rs}\partial_r q\partial_s q - 4\partial_r\,{}^3\tilde{g}^{rs}\partial_s q + {}^3\tilde{R}[r_{\bar{a}}]] =\\
&= e^{-2q}[e^{-q/2}(-8\tilde{\triangle}[r_{\bar{a}}]e^{q/2} + 8\,{}^3\tilde{\Gamma}^r_{rs}\,{}^3\tilde{g}^{su}\partial_u e^{q/2}) + {}^3\tilde{R}[r_{\bar{a}}]] =\\
&= \phi^{-5}[-8\tilde{\triangle}[r_{\bar{a}}]\phi + {}^3\tilde{R}[r_{\bar{a}}]\phi],
\end{aligned} \tag{42}$$

where ${}^3\tilde{R} = {}^3\tilde{R}[r_{\bar{a}}]$ and $\tilde{\triangle} = \tilde{\triangle}[r_{\bar{a}}]$ are the scalar curvature and the Laplace-Beltrami operator associated with the 3-metric ${}^3\tilde{g}_{rs}$ respectively [$\tilde{\triangle} - \frac{1}{8}{}^3\tilde{R}$ is a conformally invariant operator [100]]. From Eq.(D1) of Appendix D of II, we have [$\tilde{\gamma} = det\,|{}^3\tilde{g}_{rs}| = 1$]

$$\begin{aligned}
{}^3\hat{R}[q, r_{\bar{a}}] = &-\sum_{uv}\{(\partial_v q + \frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}\partial_v r_{\bar{a}})(2\partial_v q - \frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_v r_{\bar{b}}) +\\
&+ e^{-2(q + \frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}v}r_{\bar{c}})}[\partial_v^2 q + \frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}\partial_v^2 r_{\bar{a}} +\\
&+ \frac{2}{\sqrt{3}}(\partial_v q + \frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}\partial_v r_{\bar{a}})\sum_{\bar{b}}(\gamma_{\bar{b}u} - \gamma_{\bar{b}v})\partial_v r_{\bar{b}} -\\
&- (\partial_v q + \frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}v}\partial_v r_{\bar{a}})(\partial_v q + \frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_v r_{\bar{b}}]\} +\\
&+ \sum_u e^{-2(q + \frac{1}{\sqrt{3}}\sum_{\bar{c}}\gamma_{\bar{c}u}r_{\bar{c}})}[-\partial_u^2 q + \frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}\partial_u^2 r_{\bar{a}} +\\
&+ (\partial_u q + \frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}\partial_u r_{\bar{a}})(\partial_u q - \frac{2}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_u r_{\bar{b}})]\\
&\rightarrow_{r_{\bar{a}} \rightarrow 0} -6\sum_u(\partial_u q)^2 - 4e^{-2q}\sum_u[\partial_u^2 q - (\partial_u q)^2] =\\
&-24\sum_u(\partial_u ln\,\phi)^2 - 8\phi^{-4}\sum_u[\partial_u^2 ln\,\phi - 2(\partial_u ln\,\phi)^2] \rightarrow_{q\rightarrow 0} 0,
\end{aligned}$$

$$
\begin{aligned}
\rightarrow_{q\rightarrow 0}\quad & {}^3\tilde{R}[r_{\bar a}] = \\
&= -\frac{1}{\sqrt{3}}\sum_{uv}\{-\frac{1}{\sqrt{3}}\sum_{\bar a\bar b}\gamma_{\bar a u}\gamma_{\bar b u}\partial_v r_{\bar a}\partial_v r_{\bar b} + e^{-\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c v}r_{\bar c}}\sum_{\bar a}\gamma_{\bar a u}\cdot \\
&[\partial_v^2 r_{\bar a} + \frac{2}{\sqrt{3}}\sum_{\bar b}(\gamma_{\bar b u}-\gamma_{\bar b v})\partial_v r_{\bar a}\partial_v r_{\bar b} - \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b v}\partial_v r_{\bar a}\partial_v r_{\bar b}]\} + \\
&+\frac{2}{\sqrt{3}}\sum_u e^{-\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}t_{\bar c}}\sum_{\bar a}\gamma_{\bar a u}[\partial_u^2 r_{\bar a} + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_u r_{\bar a}\partial_u r_{\bar b}] = \\
&= \frac{1}{3}\sum_u(1-2e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}r_{\bar a}})\sum_{\bar b}(\partial_u r_{\bar b})^2 + \\
&+\frac{2}{\sqrt{3}}\sum_u e^{-\frac{2}{\sqrt{3}}\sum_{\bar c}\gamma_{\bar c u}t_{\bar c}}\sum_{\bar a}\gamma_{\bar a u}[\partial_u^2 r_{\bar a} + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_u r_{\bar a}\partial_u r_{\bar b}],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\triangle}[r_{\bar a}] &= \partial_r[{}^3\tilde{g}^{rs}\,\partial_s] = {}^3\tilde{g}^{rs}\,{}^3\tilde{\nabla}_r\,{}^3\tilde{\nabla}_s = \\
&= \sum_r e^{-\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\partial_r^2 - \frac{2}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_r r_{\bar b}\partial_r].
\end{aligned}
\qquad (43)
$$

Using Eq.(41), the reduced superhamiltonian constraint becomes $[k = c^3/16\pi G]$

$$
\begin{aligned}
\tilde{\mathcal{H}}_R(\tau,\vec\sigma) &= \epsilon\{k\phi^6\,{}^3\hat{R} - \frac{\phi^{-6}}{8k}{}^3G_{o(a)(b)(c)(d)}\,{}^3\hat{e}_{((a)r}\,{}^3\hat{\tilde{\pi}}^r_{(b))}\,{}^3\hat{e}_{((c)s}\,{}^3\hat{\tilde{\pi}}^s_{(d))}\}(\tau,\vec\sigma) = \\
&= \epsilon\phi(\tau,\vec\sigma)\{\,k(-8\tilde{\triangle}[r_{\bar a}] + {}^3\tilde{R}[r_{\bar a}])\phi - \\
&- \frac{\phi^{-7}}{8k}{}^3G_{o(a)(b)(c)(d)}\,{}^3\hat{e}_{((a)r}\,{}^3\hat{\tilde{\pi}}^r_{(b))}\,{}^3\hat{e}_{((c)s}\,{}^3\hat{\tilde{\pi}}^s_{(d))}\}(\tau,\vec\sigma) = \\
&= \epsilon\phi(\tau,\vec\sigma)\{\,\frac{c^3}{16\pi G}(-8\tilde{\triangle}[r_{\bar a}] + {}^3\tilde{R}[r_{\bar a}])\phi - \\
&- \frac{2\pi G}{c^3}\Big[(\phi^{-7}(6\sum_{\bar a}\pi^2_{\bar a} - \frac{1}{3}\rho^2))(\tau,\vec\sigma) + \\
&+ 2(\phi^{-5}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}r_{\bar a}}[2\sqrt{3}\sum_{\bar b}\gamma_{\bar b u}\pi_{\bar b} - \frac{1}{3}\rho])(\tau,\vec\sigma)\times \\
&\quad \int d^3\sigma_1\sum_r\delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|\phi,r_{\bar a}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1) + \\
&+ \phi^{-3}(\tau,\vec\sigma)\int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u}+r_{\bar a}(\tau,\vec\sigma)}\times \\
&\quad \sum_r\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|\phi,r_{\bar a}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1)\times \\
&\quad \sum_s\mathcal{T}^u_{(a)s}(\vec\sigma,\vec\sigma_2,\tau|\phi,r_{\bar a}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar c}\gamma_{\bar c s}\pi_{\bar c}]\Big)(\tau,\vec\sigma_2) + \\
&+ \sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a u}+\gamma_{\bar a v})r_{\bar a}(\tau,\vec\sigma)}(\delta^u_{(b)}\delta^v_{(a)} - \delta^u_{(a)}\delta^v_{(b)})\times \\
&\quad \sum_r\mathcal{T}^u_{(a)r}(\vec\sigma,\vec\sigma_1,\tau|\phi,r_{\bar a}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r}r_{\bar a}}[\frac{\rho}{3} + \sqrt{3}\sum_{\bar b}\gamma_{\bar b r}\pi_{\bar b}]\Big)(\tau,\vec\sigma_1)
\end{aligned}
$$

$$\sum_s \mathcal{T}^v_{(b)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}[\frac{\rho}{3}+\sqrt{3}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}]\Big)(\tau,\vec{\sigma}_2)\Big)\Big]\} \approx 0. \tag{44}$$

This equation has to be implemented with a gauge-fixing to eliminate $\rho(\tau,\vec{\sigma})$ [either ${}^3\hat{K}(\tau,\vec{\sigma}) \approx 0$ or $\rho(\tau,\vec{\sigma}) \approx 0$]. Let us remark that the maximal slicing condition ${}^3\hat{K}(\tau,\vec{\sigma}) \approx 0$ does not imply any real simplification of this expression and has the extra complications that: i) $r_{\bar{a}}$, $\pi_{\bar{a}}$ are no more canonical at the level of Dirac brackets; ii) one has to solve ${}^3\hat{K}(\tau,\vec{\sigma}) = 0$ as an integral equation in $\rho$, to eliminate this variable.

For $\rho(\tau,\vec{\sigma}) \approx 0$ we get the final reduced form of the Lichnerowicz equation

$$\begin{aligned}
(-\tilde{\triangle}[r_{\bar{a}}] + \frac{1}{8}{}^3\tilde{R}[r_{\bar{a}}])(\tau,\vec{\sigma})\phi(\tau,\vec{\sigma}) &= \frac{12\pi^2G^2}{c^6}\Big[2(\phi^{-7}\sum_{\bar{a}}\pi^2_{\bar{a}})(\tau,\vec{\sigma}) + \\
&+ 4(\phi^{-5}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}u}\pi_{\bar{b}})(\tau,\vec{\sigma}) \times \\
&\int d^3\sigma_1 \sum_r \delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) + \\
&+ \phi^{-3}(\tau,\vec{\sigma})\int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}(\tau,\vec{\sigma})} \times \\
&\sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) \times \\
&\sum_s \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}\Big)(\tau,\vec{\sigma}_2) + \\
&+ \sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}+\gamma_{\bar{a}v})r_{\bar{a}}(\tau,\vec{\sigma})}(\delta^u_{(b)}\delta^v_{(a)} - \delta^u_{(a)}\delta^v_{(b)}) \times \\
&\sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) \\
&\sum_s \mathcal{T}^v_{(b)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}\Big)(\tau,\vec{\sigma}_2)\Big)\Big].
\end{aligned} \tag{45}$$

Let us remark that, if this integro-differential equation for $\phi(\tau,\vec{\sigma}) = e^{\frac{1}{2}q(\tau,\vec{\sigma})} > 0$ admits different solutions $\phi_1[r_{\bar{a}},\pi_{\bar{a}}]$, $\phi_2[r_{\bar{a}},\pi_{\bar{a}}]$ ,..., they would correspond to inequivalent gravitational fields in vacuum (there are no more gauge transformations for correlating them) evolving according to the associated ADM energies.

However, if, as said in Section V of II, the existence and unicity of solutions of the 5 equations of ADM metric gravity [the Lichnerowicz equation or superhamiltonian constraint, the 3 supermomentum constraints and the gauge fixing (maximal slicing condition) ${}^3K(\tau,\vec{\sigma}) \approx 0$] remain valid in tetrad gravity for the reduced Lichnerowicz equation [the original one with the solution of the supermomentum constraints inserted into it] with the gauge fixing ${}^3K(\tau,\vec{\sigma}) \approx 0$ replaced by the natural one $\rho(\tau,\vec{\sigma}) \approx 0$ for the 3-orthogonal gauge, only one solution $\phi \approx \phi[r_{\bar{a}},\pi_{\bar{a}}]$ will exist with the boundary condition $\phi(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} 1 + \frac{M}{2r} + o_4(r^{-3/2})$ of Eqs.(37).

Finally, let us restrict ourselves to WSW hypersurfaces with the gauge-fixing procedure of Section III $[b^{(\mu)}_{(\infty)A}(\tau) \approx L^{(\mu)}{}_{(\nu)=A}(p_{(\infty)},\overset{\circ}{p}_{(\infty)})]$, see Eq.(28). The Dirac Hamiltonian is now $[\epsilon_{(\infty)} = \sqrt{\epsilon p^2_{(\infty)}}]$

$$\hat{H}^{(WSW)}_{(D)ADM,R} = \int d^3\sigma [n\hat{\mathcal{H}}_R + \lambda_n \tilde{\pi}^n](\tau,\vec{\sigma}) - \tilde{\lambda}_\tau(\tau)[\epsilon_{(\infty)} - \hat{P}^\tau_{ADM,R}] - \tilde{\lambda}_r(\tau)\hat{P}^r_{ADM,R}, \qquad (46)$$

and we have (see Eqs.(39) for the expression of $\hat{n}_r$)

$$\begin{aligned}
&N(\tau,\vec{\sigma}) = -\tilde{\lambda}_\tau(\tau) + n(\tau,\vec{\sigma}),\\
&N_{(a)}(\tau,\vec{\sigma}) = {}^3\hat{e}^r_{(a)}(\tau,\vec{\sigma})\Big[-\tilde{\lambda}_r(\tau) + \hat{n}_r[\tau,\vec{\sigma}|q,\rho,r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A]\Big] =\\
&= -\phi^{-2}(\tau,\vec{\sigma})\delta^r_{(a)}\Big[\tilde{\lambda}_r(\tau) - \hat{n}_r[\tau,\vec{\sigma}|q,\rho,r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A]\Big],
\end{aligned}$$

If we add the natural gauge-fixing $\rho(\tau,\vec{\sigma}) = \frac{1}{2}\phi(\tau,\vec{\sigma})\pi_\phi(\tau,\vec{\sigma}) \approx 0$ to $\hat{\mathcal{H}}_R(\tau,\vec{\sigma}) \approx 0$, its time constancy implies

$$\begin{aligned}
\partial_\tau \rho(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \{\rho(\tau,\vec{\sigma}), \hat{H}^{(WSW)}_{(D)ADM,R}\} = \int d^3\sigma_1 n(\tau,\vec{\sigma}_1)\{\rho(\tau,\vec{\sigma}), \hat{\mathcal{H}}_R(\tau,\vec{\sigma}_1)\} +\\
&+ \tilde{\lambda}_\tau(\tau)\{\rho(\tau,\vec{\sigma}), \hat{P}^\tau_{ADM,R}\} + \tilde{\lambda}_r(\tau)\{\rho(\tau,\vec{\sigma}), \hat{P}^r_{ADM,R}\} \approx\\
&\approx -\frac{1}{2}\phi(\tau,\vec{\sigma})\Big[\int d^3\sigma_1 n(\tau,\vec{\sigma}_1)\frac{\delta\hat{\mathcal{H}}_R(\tau,\vec{\sigma}_1)}{\delta\phi(\tau,\vec{\sigma})} +\\
&+ \tilde{\lambda}_\tau \frac{\delta\hat{P}^\tau_{ADM,R}}{\delta\phi(\tau,\vec{\sigma})} + \tilde{\lambda}_r(\tau)\frac{\delta\hat{P}^r_{ADM,R}}{\delta\phi(\tau,\vec{\sigma})} \approx 0,\\
\Rightarrow &\qquad n(\tau,\vec{\sigma}) - \hat{n}(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A] \approx 0,\\
\partial_\tau &\quad \Big[n(\tau,\vec{\sigma}) - \hat{n}(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A]\Big] =\\
&= \lambda_n(\tau,\vec{\sigma}) - \{\hat{n}(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A], \hat{H}^{(WSW)}_{(D)ADM,R}\} \approx 0,\\
\Rightarrow &\qquad \lambda_n(\tau,\vec{\sigma}) \quad \textit{determined.}
\end{aligned} \qquad (47)$$

Therefore we find an integral equation for the lapse function $n(\tau,\vec{\sigma})$ implying its being different from zero (this avoids a finite time breakdown), even when $\vec{\tilde{\lambda}}(\tau) = 0$ since $\tilde{\lambda}_\tau(\tau) \neq 0$.

If we now go to the final Dirac brackets with respect to the second class constraints $\rho(\tau,\vec{\sigma}) \approx 0$, $\hat{\mathcal{H}}_R(\tau,\vec{\sigma}) \approx 0$, $n(\tau,\vec{\sigma}) - \hat{n}(\tau,\vec{\sigma}|r_{\bar{a}},\pi_{\bar{a}},\tilde{\lambda}_A] \approx 0$, $\tilde{\pi}^n(\tau,\vec{\sigma}) \approx 0$, on the WSW hypersurfaces, asymptotically orthogonal to $\hat{P}^{(\mu)}_{ADM,R}$ at spatial infinity, we remain only with the canonical variables $r_{\bar{a}}$, $\pi_{\bar{a}}$ from Eqs.(28) we get the following form of the Dirac-Hamiltonian and of the remaining four first class constraints

$$\hat{H}^{(WSW)}_{(D)ADM,R\rho=0} = -\tilde{\lambda}_\tau(\tau)\Big(\epsilon_{(\infty)} - \hat{P}^\tau_{ADM,R}[r_{\bar{a}},\pi_{\bar{a}},\phi(r_{\bar{a}},\pi_{\bar{a}})]\Big) - \tilde{\lambda}_r(\tau)\hat{P}^r_{ADM,R}[r_{\bar{a}},\pi_{\bar{a}},\phi(r_{\bar{a}},\pi_{\bar{a}})],$$

$$\begin{aligned}
&\epsilon_{(\infty)} - \hat{P}^\tau_{ADM,R}[r_{\bar{a}},\pi_{\bar{a}},\phi(r_{\bar{a}},\pi_{\bar{a}})] \approx 0,\\
&\hat{P}^r_{ADM,R}[r_{\bar{a}},\pi_{\bar{a}},\phi(r_{\bar{a}},\pi_{\bar{a}})] \approx 0.
\end{aligned} \qquad (48)$$

where $\phi(r_{\bar{a}},\pi_{\bar{a}})$ is the solution of the reduced Lichnerowicz equation $\hat{\mathcal{H}}_R(\tau,\vec{\sigma})|_{\rho(\tau,\vec{\sigma})=0} = 0$. The Dirac Hamiltonian takes the same form as in the case of parametrized Minkowski theories restricted to Wigner spacelike hyperplanes.

After the gauge-fixing $T_{(\infty)} - \tau \approx 0$, one gets $\tilde{\lambda}_\tau(\tau) = \epsilon$ and

$$\begin{aligned} N(\tau, \vec{\sigma}) &= -\epsilon + \hat{n}(\tau, \vec{\sigma}|r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}}), \tilde{\lambda}_r] = \\ &= -\epsilon + \tilde{n}(\tau, \vec{\sigma}|r_{\bar{a}}, \pi_{\bar{a}}, \tilde{\lambda}_r], \\ N_r(\tau, \vec{\sigma}) &= -\tilde{\lambda}_r(\tau) + \hat{n}_r(\tau, \vec{\sigma}|r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}}), \tilde{\lambda}_r] = \\ &= -\tilde{\lambda}_r(\tau) + \tilde{n}_r(\tau, \vec{\sigma}|r_{\bar{a}}, \pi_{\bar{a}}, \tilde{\lambda}_s], \end{aligned} \tag{49}$$

$$\hat{H}^{(WSW)}_{(D)ADM} = \epsilon \hat{P}^\tau_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}})] - \tilde{\lambda}_r(\tau) \hat{P}^r_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}})],$$

$$\hat{P}^r_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}})] \approx 0. \tag{50}$$

This is the asymptotic rest-frame instant form of dynamics for tetrad gravity.

In the gauge $\vec{\tilde{\lambda}}(\tau) = 0$, implied by the gauge fixings $\vec{\sigma}_{ADM}[r_{\bar{a}}, \pi_{\bar{a}}] \approx 0$ on the "internal" 3-center-of-mass, we get the final Dirac Hamiltonian

$$\hat{H}^{(WSW)'}_{(D)ADM} = \epsilon \hat{P}^\tau_{ADM,R},$$

and the following normal form [namely solved in the accelerations] of the two dynamical Einstein equations for the gravitational field Dirac observables in the 3-orthogonal gauge with $\rho(\tau, \vec{\sigma}) \approx 0$ and in the rest frame

$$\begin{aligned} \partial_\tau r_{\bar{a}}(\tau, \vec{\sigma}) &\, {\buildrel \circ \over =}\, \{ r_{\bar{a}}(\tau, \vec{\sigma}), \hat{P}^\tau_{ADM,R}[r_{\bar{b}}, \pi_{\bar{b}}, \phi(r_{\bar{b}}, \pi_{\bar{b}})] \}, \\ \partial_\tau \pi_{\bar{a}}(\tau, \vec{\sigma}) &\, {\buildrel \circ \over =}\, \{ \pi_{\bar{a}}(\tau, \vec{\sigma}), \hat{P}^\tau_{ADM,R}[r_{\bar{b}}, \pi_{\bar{b}}, \phi(r_{\bar{b}}, \pi_{\bar{b}})] \}, \end{aligned}$$

$$\hat{P}^r_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}})] \approx 0. \tag{51}$$

The 4-metric and the line element in adapted coordinates $\sigma^A$ on the WSW hypersurfaces are

$$\begin{aligned} {}^4\hat{g}_{AB} &= \\ &= \epsilon \begin{pmatrix} \Big( - \epsilon + \tilde{n}[r_{\bar{a}}, \pi_{\bar{a}}, \vec{\tilde{\lambda}}] \Big)^2 - \phi^{-4}[r_{\bar{a}}, \pi_{\bar{a}}] \sum_r e^{-\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \Big( \tilde{\lambda}^2_r(\tau) - \tilde{n}_r[r_{\bar{a}}, \pi_{\bar{a}}, \vec{\tilde{\lambda}} \Big)^2 & \tilde{\lambda}_r(\tau) - \tilde{n}_r[r_{\bar{a}}, \pi_{\bar{a}}, \vec{\tilde{\lambda}}] \\ \tilde{\lambda}_s(\tau) - \tilde{n}_s[r_{\bar{a}}, \pi_{\bar{a}}, \vec{\tilde{\lambda}}] & -[\phi^4[r_{\bar{a}}, \pi_{\bar{a}}] e^{\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}}] \delta_{rs} \end{pmatrix} \\ &\rightarrow_{\vec{\tilde{\lambda}}(\tau, \vec{\sigma}) = 0} \epsilon \begin{pmatrix} \Big( - \epsilon + \tilde{n}[r_{\bar{a}}, \pi_{\bar{a}}, 0] \Big)^2 - \phi^{-4}[r_{\bar{a}}, \pi_{\bar{a}}] \sum_r e^{-\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \tilde{n}^2_r[r_{\bar{a}}, \pi_{\bar{a}}, 0] & -\tilde{n}_r[r_{\bar{a}}, \pi_{\bar{a}}, 0] \\ -\tilde{n}_s[r_{\bar{a}}, \pi_{\bar{a}}, 0] & -\phi^4[r_{\bar{a}}, \pi_{\bar{a}}] e^{\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \delta_{rs} \end{pmatrix}, \end{aligned}$$

$$ds^2 = {}^4\hat{g}_{\tau\tau} (d\tau)^2 + 2\, {}^4\hat{g}_{\tau r} d\tau d\sigma^r + \sum_r {}^4\hat{g}_{rr} (d\sigma^r)^2. \tag{52}$$

Even for $\vec{\tilde{\lambda}}(\tau)=0$ we do not get vanishing shift functions ["synchronous" coordinates], like instead it is assumed by Christodoulou and Klainermann for their singularity-free solutions.

Let us remark that Eqs.(39) and (47) imply that both $\tilde{n}$ and $\tilde{n}_r$ depend on $G/c^3$ and $c^3/G$ simultaneously, so that both their post-Newtonian (expansion in $1/c$) and post-Minkowskian (formal expansion in powers of G) may be non trivial after having done the gauge fixings.

Once we are on WSW hypersurfaces, each triad ${}^3e^r_{(a)}(\tau,\vec{\sigma})$ has an asymptotic limit, which is also the limit of one of the Frauendiener triads ${}^3e^{(W)r}_{(a)}(\tau,\vec{\sigma})$, solutions of Eqs.(32). In Eqs.(32), ${}^3\hat{K}(\tau,\vec{\sigma})$ is the function of $r_{\bar{a}}$, $\pi_{\bar{a}}$, $q$ [for $\rho(\tau,\vec{\sigma})\approx 0$] determined by Eq.(38), so that also the triads ${}^3e^{(W)r}_{(a)}(\tau,\vec{\sigma})$ become functionals of these same variables.

From Eqs.(40) evaluated with the gauge fixing $\rho(\tau,\vec{\sigma})\approx 0$, we get the weak ADM Poincaré charges in this gauge

$$
\begin{aligned}
\hat{P}^\tau_{ADM,R} &= \epsilon\int d^3\sigma\Big(\frac{c^3}{16\pi G}\big[\phi^2\sum_r e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\times\\
&\quad \Big(\frac{1}{3}\sum_{\bar{b}\bar{c}}(2\gamma_{\bar{b}r}\gamma_{\bar{c}r}+\delta_{\bar{b}\bar{c}})\partial_r r_{\bar{b}}\partial_r r_{\bar{c}} - \frac{4}{\sqrt{3}}(\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_r r_{\bar{b}})\partial_r ln\,\phi - 4(\partial_r ln\,\phi)^2 -\\
&- \sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}r}-\gamma_{\bar{a}u})r_{\bar{a}}}[\frac{2}{3}\sum_{\bar{b}\bar{c}}\gamma_{\bar{b}r}\gamma_{\bar{c}r}\partial_u r_{\bar{b}}\partial_u r_{\bar{c}} +\\
&+ 2\sqrt{3}(\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_u r_{\bar{b}})\partial_u ln\,\phi + 4(\partial_u ln\,\phi)^2]\Big)\big](\tau,\vec{\sigma}) -\\
&- \frac{6\pi G}{c^3}\phi^{-2}(\tau,\vec{\sigma})\Big[(2\phi^{-4}\sum_{\bar{a}}\pi^2_{\bar{a}})(\tau,\vec{\sigma}) +\\
&+ 4(\phi^{-2}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}u}\pi_{\bar{b}})(\tau,\vec{\sigma})\times\\
&\quad \int d^3\sigma_1\sum_r \delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) +\\
&+ \int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}(\tau,\vec{\sigma})}\times\\
&\quad \sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1)\times\\
&\quad \sum_s \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}\Big)(\tau,\vec{\sigma}_2) +\\
&+ \sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}+\gamma_{\bar{a}v})r_{\bar{a}}(\tau,\vec{\sigma})}(\delta^u_{(b)}\delta^v_{(a)} - \delta^u_{(a)}\delta^v_{(b)})\times\\
&\quad \sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1)\\
&\quad \sum_s \mathcal{T}^v_{(b)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}\Big)(\tau,\vec{\sigma}_2)\Big)\Big]\Big),\\
\hat{P}^r_{ADM,R} &= \sqrt{3}\int d^3\sigma\phi^{-2}(\tau,\vec{\sigma})e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})}\\
&\quad \Big(\phi^{-2}(\tau,\vec{\sigma})\sum_{\bar{c}}\Big[\sum_u\gamma_{\bar{c}u}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}(2\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}u}\partial_r r_{\bar{b}}) -
\end{aligned}
$$

$$
\begin{aligned}
&- 2\gamma_{\bar c r} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} (2\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_r r_{\bar b})\Big](\tau,\vec\sigma)\pi_{\bar c}(\tau,\vec\sigma) + \\
&+ \sum_{uv}\int d^3\sigma_1 \Big[(2\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_r r_{\bar b})(\tau,\vec\sigma)\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec\sigma,\vec\sigma_1,\tau|\phi, r_{\bar a}] - \\
&- (2\partial_u ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b})(\tau,\vec\sigma)\Big(\delta_{r(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^r_{(a)v}\Big)(\vec\sigma,\vec\sigma_1,\tau|\phi, r_{\bar a}]\Big] \\
&\quad (\phi^{-2} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v} r_{\bar a}} \sum_{\bar c}\gamma_{\bar c v}\pi_{\bar c})(\tau,\vec\sigma_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\hat J^{rs}_{ADM,R} &= -\frac{\sqrt{3}}{2}\int d^3\sigma\phi^{-4}(\tau,\vec\sigma)\sum_{\bar c}\pi_{\bar c}(\tau,\vec\sigma) \\
&\quad \Big[\sigma^s e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}}\Big(\sum_u \gamma_{\bar c u} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}}(2\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_r r_{\bar b}) - \\
&- 2\gamma_{\bar c r} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}}(2\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_r r_{\bar b})\Big) - \\
&- \sigma^r e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s} r_{\bar a}}\Big(\sum_u \gamma_{\bar c u} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a u} r_{\bar a}}(2\partial_s ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_s r_{\bar b}) - \\
&- 2\gamma_{\bar c s} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s} r_{\bar a}}(2\partial_s ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_s r_{\bar b})\Big)\Big](\tau,\vec\sigma) + \\
&+ \frac{\sqrt{3}}{2}\sum_{uv}\int d^3\sigma d^3\sigma_1 \phi^{-2}(\tau,\vec\sigma) \\
&\quad \Big[\sigma^s e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}(\tau,\vec\sigma)}\Big((2\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_r r_{\bar b})(\tau,\vec\sigma)\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec\sigma,\vec\sigma_1,\tau|\phi, r_{\bar a}] - \\
&- (2\partial_u ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b})(\tau,\vec\sigma)\Big(\delta_{r(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^r_{(a)v}\Big)(\vec\sigma,\vec\sigma_1,\tau|\phi, r_{\bar a}]\Big) - \\
&- \sigma^r e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a s} r_{\bar a}(\tau,\vec\sigma)}\Big((2\partial_s ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b u}\partial_s r_{\bar b})(\tau,\vec\sigma)\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec\sigma,\vec\sigma_1,\tau|\phi, r_{\bar a}] - \\
&- (2\partial_u ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar b}\gamma_{\bar b s}\partial_u r_{\bar b})(\tau,\vec\sigma)\Big(\delta_{s(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^s_{(a)v}\Big)(\vec\sigma,\vec\sigma_1,\tau|\phi, r_{\bar a}]\Big)\Big] \\
&\quad (\phi^{-2} e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a v} r_{\bar a}} \sum_{\bar c}\gamma_{\bar c v}\pi_{\bar c})(\tau,\vec\sigma_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\hat J^{\tau r}_{ADM,R} &= \epsilon\int d^3\sigma\sigma^r\Big(\frac{c^3}{16\pi G}\Big[\phi^2\sum_r e^{-\frac{1}{\sqrt{3}}\sum_{\bar a}\gamma_{\bar a r} r_{\bar a}} \times \\
&\quad \Big(\frac{1}{3}\sum_{\bar b\bar c}(2\gamma_{\bar b r}\gamma_{\bar c r} + \delta_{\bar b\bar c})\partial_r r_{\bar b}\partial_r r_{\bar c} - \frac{4}{\sqrt{3}}(\sum_{\bar b}\gamma_{\bar b r}\partial_r r_{\bar b})\partial_r ln\,\phi - 4(\partial_r ln\,\phi)^2 - \\
&- \sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar a}(\gamma_{\bar a r} - \gamma_{\bar a u}) r_{\bar a}}[\frac{2}{3}\sum_{\bar b\bar c}\gamma_{\bar b r}\gamma_{\bar c r}\partial_u r_{\bar b}\partial_u r_{\bar c} + \\
&+ 2\sqrt{3}(\sum_{\bar b}\gamma_{\bar b r}\partial_u r_{\bar b})\partial_u ln\,\phi + 4(\partial_u ln\,\phi)^2]\Big)\Big](\tau,\vec\sigma) -
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{6\pi G}{c^3}\phi^{-2}(\tau,\vec{\sigma})\Big[(2\phi^{-4}\sum_{\bar{a}}\pi^2_{\bar{a}})(\tau,\vec{\sigma}) + \\
&+ 4(\phi^{-2}\sum_u e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}u}\pi_{\bar{b}})(\tau,\vec{\sigma}) \times \\
&\quad \int d^3\sigma_1 \sum_r \delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) + \\
&+ \int d^3\sigma_1 d^3\sigma_2\Big(\sum_u e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}u}r_{\bar{a}}(\tau,\vec{\sigma})} \times \\
&\quad \sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) \times \\
&\quad \sum_s \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}\Big)(\tau,\vec{\sigma}_2) + \\
&+ \sum_{uv} e^{\frac{1}{\sqrt{3}}\sum_{\bar{a}}(\gamma_{\bar{a}u}+\gamma_{\bar{a}v})r_{\bar{a}}(\tau,\vec{\sigma})}(\delta^u_{(b)}\delta^v_{(a)} - \delta^u_{(a)}\delta^v_{(b)}) \times \\
&\quad \sum_r \mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\sum_{\bar{b}}\gamma_{\bar{b}r}\pi_{\bar{b}}\Big)(\tau,\vec{\sigma}_1) \\
&\quad \sum_s \mathcal{T}^v_{(b)s}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,r_{\bar{a}}]\Big(\phi^{-2}e^{-\frac{1}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}s}r_{\bar{a}}}\sum_{\bar{c}}\gamma_{\bar{c}s}\pi_{\bar{c}}\Big)(\tau,\vec{\sigma}_2)\Big)\Big]\Big) + \\
&+ \frac{\epsilon c^3}{8\pi G}\int d^3\sigma\phi^{-2}(\tau,\vec{\sigma})e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}(\tau,\vec{\sigma})} \\
&\quad \Big[\sum_v e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}v}r_{\bar{a}}}\Big(\phi^4 e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}v}r_{\bar{a}}} - 1\Big)(\partial_r ln\,\phi + \frac{1}{\sqrt{3}}\sum_{\bar{b}}(\gamma_{\bar{b}r}+\gamma_{\bar{b}v})\partial_r r_{\bar{b}}) - \\
&- e^{-\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}}\Big(\phi^4 e^{\frac{2}{\sqrt{3}}\sum_{\bar{a}}\gamma_{\bar{a}r}r_{\bar{a}}} - 1\Big)(\partial_r ln\,\phi + \frac{2}{\sqrt{3}}\sum_{\bar{b}}\gamma_{\bar{b}r}\partial_r r_{\bar{b}})\Big](\tau,\vec{\sigma}). \qquad (53)
\end{aligned}
$$

From Eq.(45) it is clear that $\phi[r_{\bar{a}},\pi_{\bar{a}}]$ depends on $G^2/c^6$, while the previous equation implies that $\hat{H}^{(WSW)'}_{(D)ADM} = \epsilon\hat{P}^\tau_{ADM,R}$ depends a priori on both $G/c^3$ and $c^3/G$.

Let us remark that to try to make realistic calculations one needs normal coordinates around a point (see Appendix A of II): only in such a chart one can find the explicit expression of the Synge-DeWitt bitensor $d^r_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)$ (giving the tangent in $P_1$ $[\vec{\sigma}_1]$ of the geodesic emanating from P $[\vec{\sigma}]$).

A fundamental open problem, which will be studied in a future paper, is to find a solution of a linearization of the Lichnerowicz equation, which put inside a linearization of the weak ADM energy will imply a linear equation for the physical canonical variables $r_{\bar{a}}(\tau,\vec{\sigma})$ describing the gravitational field, so to make contact with the theory of gravitational waves.

A connected open problem is to find the relation of our canonical variables $r_{\bar{a}}$, $\pi_{\bar{a}}$ for the gravitational field in the special 3-orthogonal gauge with $\rho = 0$ with the statement of Christodoulou- Klainermann [7] that the independent degrees of freedom of the gravitational field are described by symmetric trace-free 2-tensors on 2-planes. See the Conclusions and Appendix F for some comments on this point.

Finally in Appendix G there are some comments on the post-Newtonian approximation.

## VI. VOID SPACETIMES IN THE 3-ORTHOGONAL GAUGE.

As said in II it is interesting to study the tiny subspace of the reduced phase space in the 3-orthogonal gauges defined by putting by hand equal to zero the Dirac's observables of the gravitational field $[r_{\bar{a}}(\tau,\vec{\sigma}) \approx 0,\, \pi_{\bar{a}}(\tau,\vec{\sigma}) \approx 0]$ before adding the gauge-fixing $\rho(\tau,\vec{\sigma}) \approx 0$, since in absence of matter this is consistent with Einstein equations. These spacetimes may be named "void spacetimes" and it will be shown in this Section that they are gauge equivalent to Minkowski spacetime in Cartesian coordinates by means of the Hamiltonian group of gauge transformations and that they have vanishing Poincaré charges. They should correspond to the relativistic generalization of the class of Galilean non inertial frames (with their inertial forces) obtainable from an inertial frame of the nonrelativistic Galileo spacetime [for example the (maybe time-dependent) pseudo-diffeomorphisms in $Diff\, \Sigma_\tau$ replace the Galilean coordinate transformations generating the inertial forces].

The concept of void spacetime implements the viewpoint of Synge [102] that, due to tidal (i.e. curvature) effects, there is a difference between true gravitational fields and accelerated motions, even if, as shown in Ref. [103], Einstein arrived at general relativity through the intermediate step of showing the equivalence of uniform acceleration with special homogeneous gravitational fields. It is only in the not generally covariant Hamiltonian approach that one is able to identify the genuine physical degrees of freedom of the gravitational field.

Since in void spacetimes without matter there are no physical degrees of freedom of the gravitational field but only gauge degrees of freedom, we expect that this equivalence class of spacetimes is not described by scenario b) of Section III, but that it corresponds to scenario a) with vanishing Poincaré charges (the exceptional Poincaré orbit). Indeed, in this way Minkowski spacetime (and its gauge copies) would be selected as the static background for special relativity with zero energy (in contrast with the viewpoint of Ref. [42] of an infinite background energy), starting point for parametrized Minkowski theories where the special relativistic energy would be generated only by the added matter (and/or fields). Tetrad gravity with matter would be described by scenario b) (with the WSW hypersurfaces corresponding to Wigner hyperplanes; both of them are defined only in presence of matter) and in the limit $G \rightarrow 0$ the weak ADM energy would tend to the special relativistic energy of that matter system with no trace left of the "gravitational field energy" [present in scenario b) but not in scenario a)].

To define "void spacetimes" independently from the 3-orthogonal gauge, let us remark that, since the conditions $r_{\bar{a}}(\tau,\vec{\sigma}) = 0$ imply the vanishing of the 3-conformal Cotton-York tensor [see I after Eq.(9) for the definition of this tensor and Eq.(D2) of Appendix D of II for its vanishing], this means that void spacetimes should have the leaves $\Sigma_\tau$ conformally flat as Riemannian 3-manifolds but with the conformal factor determined by the reduced Lichnerowicz equation (since the solution depends on the gauge variable $\rho$, the conformal factor is gauge dependent). Therefore, the general theory of void spacetimes could be reformulated in arbitrary gauges by adding with Lagrange multipliers the two independent components of the Cotton-York tensor ${}^3\mathcal{Y}_{rs}(\tau,\vec{\sigma})$ [which is a function only of cotriads] to the tetrad ADM Lagrangian of Eq.(50) of I for tetrad gravity. In this way one should get two extra holonomic constraints equivalent to $r_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$. Their time constancy should produce two secondary (momentum dependent) constraints equivalent to $\pi_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$.

Deferring to a future paper the study of the general case, let us explore the properties

of void spacetimes in the 3-orthogonal gauges.

To get void spacetimes starting from the 3-orthogonal gauge definition of the gravitational field degrees of freedom, we add with Dirac's multipliers to the tetrad gravity version of the Dirac Hamiltonian of Eq(25) the two pairs of primary second class constraints $r_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$, $\pi_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$:

$\hat{H}^{(1)}_{(D)ADM} \mapsto \hat{H}^{(1)}_{(D)ADM} + \int d^3\sigma \sum_{\bar{a}} [\xi_{\bar{a}} r_{\bar{a}} + \xi^{'}_{\bar{a}} \pi_{\bar{a}}](\tau\vec{\sigma})$.

The time constancy of these constraints determines the Dirac multipliers $\xi_{\bar{a}}(\tau,\vec{\sigma})$, $\xi^{'}_{\bar{a}}(\tau,\vec{\sigma})$. By going to new Dirac brackets $[r_{\bar{a}}(\tau,\vec{\sigma}) \equiv \pi_{\bar{a}}(\tau,\vec{\sigma}) \equiv 0]$ we get the 3-metric ${}^3\hat{g}_{rs} = e^{2q}\delta_{rs} = \phi^4\delta_{rs}$ and we remain only with the following variables:
i) $\phi(\tau,\vec{\sigma}) = e^{q(\tau,\vec{\sigma})/2}$, to be determined by the reduced Lichnerowicz equation;
ii) $\pi_\phi(\tau,\vec{\sigma}) = 2\phi^{-1}(\tau,\vec{\sigma})\rho(\tau,\vec{\sigma})$, the conjugate gauge variable.

In this way we have identified some members [they differ in the arbitrary value of $\pi_\phi = 2\phi^{-1}\rho$] of a Hamiltonian equivalence class, which corresponds to the absence of the gravitational field. For it we have

$$\begin{aligned}
&r_{\bar{a}} = \pi_{\bar{a}} = 0, \qquad \phi = e^{q/2} \quad [or\, q = 2ln\, \phi],\\
&{}^3\hat{e}_{(a)r} = \phi^2\delta_{(a)r}, \qquad {}^3\hat{e}^r_{(a)} = \phi^{-2}\delta^r_{(a)},\\
&{}^3\hat{g}_{rs} = \phi^4\delta_{rs}, \qquad {}^3\tilde{g}_{rs}[r_{\bar{a}} = 0] = \delta_{rs},\\
&\tilde{\triangle}[r_{\bar{a}} = 0] = \triangle_{FLAT}, \qquad {}^3\tilde{R}[r_{\bar{a}} = 0] = 0.
\end{aligned} \tag{54}$$

From Eqs.(37), (38), one has in void spacetimes [before putting $\rho = \frac{1}{2}\phi\pi_\phi = 0$]

$$\begin{aligned}
{}^3\hat{g}_{rs}(\tau,\vec{\sigma}) &= = \phi^4(\tau,\vec{\sigma})\delta_{rs},\\
{}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) &= \frac{1}{3}\int d^3\sigma_1 \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,0]\rho(\tau,\vec{\sigma}_1),
\end{aligned}$$

$$\mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0] = \delta^r_{(a)}\delta^r_s\delta^3(\vec{\sigma},\vec{\sigma}_1) + \mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0],$$

$$\begin{aligned}
\mathcal{T}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,0] = \frac{1}{2}[2\sum_{w\neq s}\delta_{(k)w}\frac{\partial ln\,\phi(\tau,\vec{\sigma}_1)}{\partial\sigma_1^w}\phi^{-2}(\tau,\vec{\sigma})\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) +\\
+\delta_{(k)s}\frac{\partial}{\partial\sigma_1^s}\phi^{-2}(\tau,\vec{\sigma})\delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau)],
\end{aligned}$$

$$\begin{aligned}
\phi^{-2}(\tau,\vec{\sigma}) \quad & \delta^r_{(b)}T_{(b)(a)(k)}(\vec{\sigma},\vec{\sigma}_1;\tau) =\\
&= d^r_{\gamma_{PP_1}}(P_{\gamma_{PP_1}}\, e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}} d\sigma_2^w\, {}^3\hat{\omega}_{w(c)}(\tau,\vec{\sigma}_2)\hat{R}^{(c)}})_{(a)(k)} +\\
&+ \sum_u \delta_{(a)u}d^u_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)\\
&\quad \delta^r_{(b)}(P_{\gamma_{PP_1}}\, e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}} d\sigma_2^w\, {}^3\hat{\omega}_{w(c)}(\tau,\vec{\sigma}_2)\hat{R}^{(c)}})_{(b)(k)},
\end{aligned}$$

$${}^3\hat{\omega}_{t(d)}(\tau,\vec{\sigma}) = 2\epsilon_{(d)(m)(n)}\delta_{(m)t}\delta_{(n)u}\partial_u ln\,\phi(\tau,\vec{\sigma}). \tag{55}$$

and

$$
\begin{aligned}
{}^3\hat{K}_{rs}(\tau,\vec{\sigma}) &= \frac{\epsilon}{4k}[\sum_u(\delta_{ru}\delta_{(a)s}+\delta_{su}\delta_{(a)r}-\delta_{rs}\delta_{(a)u})\,{}^3\hat{\tilde{\pi}}^u_{(a)}](\tau,\vec{\sigma}),\\
{}^3\hat{K}(\tau,\vec{\sigma}) &= -\frac{\epsilon}{4k}[\phi^{-4}\sum_u\delta_{(a)u}\,{}^3\hat{\tilde{\pi}}^u_{(a)}](\tau,\vec{\sigma}) =\\
&= -\frac{\epsilon}{4k}\phi^{-6}(\tau,\vec{\sigma})\{\rho(\tau,\vec{\sigma})+\frac{1}{3}\sum_u\int d^3\sigma_1\delta_{(a)u}\\
&\quad \mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,0]\phi^{-2}(\tau,\vec{\sigma}_1)\rho(\tau,\vec{\sigma}_1)\},
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{H}}^{'}_R(\tau,\vec{\sigma}) &= \hat{\mathcal{H}}_R(\tau,\vec{\sigma})|_{r_{\bar{a}}=\pi_{\bar{a}}=0} =\\
&= \epsilon\phi(\tau,\vec{\sigma})\{-\frac{c^3}{2\pi G}\triangle_{FLAT}\phi(\tau,\vec{\sigma})+\frac{2\pi G}{3c^3}\Big[\frac{1}{3}(\phi^{-7}\rho)(\tau,\vec{\sigma})+\\
&+\frac{2}{3}(\phi^{-5}\rho)(\tau,\vec{\sigma})\int d^3\sigma_1\sum_r\delta^u_{(a)}\mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,0](\phi^{-2}\rho(\tau,\vec{\sigma}_1)-\\
&-\frac{1}{3}\phi^{-3}(\tau,\vec{\sigma})\int d^3\sigma_1 d^3\sigma_2\Big(\sum_u\\
&\quad \sum_r\mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)\\
&\quad \sum_s\mathcal{T}^u_{(a)s}(\vec{\sigma},\vec{\sigma}_2;\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_2)+\\
&+\sum_{uv}(\delta^u_{(b)}\delta^v_{(a)}-\delta^u_{(a)}\delta^v_{(b)})\\
&\quad \sum_r\mathcal{T}^u_{(a)r}(\vec{\sigma},\vec{\sigma}_1;\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)\\
&\quad \sum_s\mathcal{T}^v_{(a)s}(\vec{\sigma},\vec{\sigma}_2;\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_2)\Big)\Big]\}\approx 0.
\end{aligned}
\tag{56}
$$

Therefore, the leaves $\Sigma_\tau$ of the 3+1 splittings of void spacetimes are conformally flat 3-manifolds diffeomorphic to $R^3$, but with the conformal factor $\phi$ determined by the reduced Lichnerowicz equation.

With the natural gauge $\rho(\tau,\vec{\sigma})\approx 0$, one has

$$
\begin{aligned}
&{}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma})\approx 0,\\
&{}^3\hat{K}_{rs}(\tau,\vec{\sigma}) \;\approx\; 0, \qquad \Rightarrow {}^3\hat{K}(\tau,\vec{\sigma})\approx 0,
\end{aligned}
\tag{57}
$$

and the reduced superhamiltonian constraint becomes the reduced Lichnerowicz equation $[\triangle_{FLAT}=\vec{\partial}^2]$

$$
\triangle_{FLAT}\phi(\tau,\vec{\sigma})\approx 0,\quad \Rightarrow \phi(\tau,\vec{\sigma})=1, \Rightarrow {}^3\hat{g}_{rs}=\delta_{rs},
\tag{58}
$$

where we have shown the solution corresponding to the boundary condition of Eq.(37).

Let us now delineate the sequence of natural gauge-fixings to get void spacetimes in the 3-orthogonal gauges from scenario a) [in which $\tilde{\lambda}_A(\tau)$, $\tilde{\lambda}_{AB}(\tau)$ are gauge variables]:

i) $\tilde{\lambda}_\tau(\tau)\approx\epsilon$, $\tilde{\lambda}_r(\tau)\approx 0$, $\tilde{\lambda}_{AB}(\tau)\approx 0$ [so that $N_{(as)}(\tau,\vec{\sigma})\approx 0$, $N_{(as)r}(\tau,\vec{\sigma})\approx 0$]: these are the gauge-fixings for the primary constraints $\tilde{\pi}^A(\tau)\approx 0$, $\tilde{\pi}^{AB}(\tau)\approx 0$ and imply $\zeta_A(\tau)\approx 0$, $\zeta_{AB}(\tau)\approx 0$.

ii) $\alpha_{(a)}(\tau,\vec{\sigma}) = \varphi_{(a)}(\tau,\vec{\sigma}) = 0$ [so that $\lambda^{\vec{\varphi}}_{(a)}(\tau,\vec{\sigma}) = \hat{\mu}_{(a)}(\tau,\vec{\sigma}) = 0$ in the Dirac Hamiltonian];
iii) $\xi^r(\tau,\vec{\sigma}) = \sigma^r$. Since, as we have seen, in void spacetimes ${}^3\tilde{\pi}^r_{(a)}$ and ${}^3\tilde{\Pi}^{rs}$ vanish for $\rho(\tau,\vec{\sigma}) \approx 0$, also the Poisson bracket $\{\xi^r(\tau,\vec{\sigma}), \hat{P}^\tau_{ADM}\}$ vanishes for $\rho(\tau,\vec{\sigma}) \approx 0$. Therefore the requirement $\partial_\tau[\xi^r(\tau,\vec{\sigma}) - \sigma^r] \approx 0$ now implies $n_r(\tau,\vec{\sigma}) - \hat{n}_r(\tau,\vec{\sigma}|q,\rho] \approx 0$ but with $\hat{n}_r(\tau,\vec{\sigma}|q,\rho]|_{\rho=0} = 0$; the time constancy of these constraints determine $\lambda^{\vec{n}}_r(\tau,\vec{\sigma})$. For $\rho(\tau,\vec{\sigma}) \approx 0$ we get a vanishing shift function $N_r(\tau,\vec{\sigma}) \approx 0$ (synchronous coordinates).
iv) At this stage the lapse function is

$$N(\tau,\vec{\sigma}) \approx -\epsilon + n(\tau,\vec{\sigma}).$$

v) Now we add the second class constraints $r_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$, $\pi_{\bar{a}}(\tau,\vec{\sigma}) \approx 0$ which imply the previous results.
vi) The Dirac Hamiltonian becomes

$$H^{(1)}_{(D)ADM} = \int d^3\sigma [n\hat{\mathcal{H}}'_R + \lambda_n \tilde{\pi}^n](\tau,\vec{\sigma}) + \epsilon \hat{P}^{\tau\,\prime}_{ADM}$$

with $\hat{\mathcal{H}}'_R = \hat{\mathcal{H}}_R|_{r_{\bar{a}}=\pi_{\bar{a}}=0}$, $\hat{P}^{\tau\,\prime}_{ADM} = \hat{P}^\tau_{ADM}|_{r_{\bar{a}}=\pi_{\bar{a}}=0}$.
vii) The natural gauge fixing $\rho(\tau,\vec{\sigma}) \approx 0$ implies

$$\partial_\tau \rho(\tau,\vec{\sigma}) \stackrel{\circ}{=} \int d^3\sigma_1 n(\tau,\vec{\sigma}_1)\{\rho(\tau,\vec{\sigma}), \hat{\mathcal{H}}'_R(\tau,\vec{\sigma}_1)\} + \epsilon\{\rho(\tau,\vec{\sigma}), \hat{P}^{\tau\,\prime}_{ADN}\};$$

but from Eq.(14) and from ${}^3\tilde{\Pi}^{rs} \approx 0$ we see that only the term bilinear in the Christoffel symbols contributes to $\{\rho(\tau,\vec{\sigma}), \hat{P}^{\tau\,\prime}_{ADN}\}$ for $\rho(\tau,\vec{\sigma}) \approx 0$. Now from Eq.(38) we get ${}^3\hat{\Gamma}^r_{uv} = 2\phi^{-1}[\delta_{uv}\partial_r\phi + \delta_{ru}\partial_v\phi + \delta_{rv}\partial_u\phi] \rightarrow_{\phi\rightarrow const.} 0$. Since $\phi(\tau,\vec{\sigma}) = 1$ is the solution of the reduced Lichnerowicz equation for $\rho(\tau,\vec{\sigma}) \approx 0$, we get $\{\rho(\tau,\vec{\sigma}), \hat{P}^{\tau\,\prime}_{ADN}\} \approx 0$ and then $n(\tau,\vec{\sigma}) \approx 0$ and $\lambda_n(\tau,\vec{\sigma}) \approx 0$. Therefore, at the end the lapse function is $N(\tau,\vec{\sigma}) \approx -\epsilon$.
viii) Since, as we shall see, $\hat{P}^{\tau\,\prime}_{ADN}$ vanishes for $\rho(\tau,\vec{\sigma}) \approx 0$, $\phi(\tau,\vec{\sigma}) = 1$, the final Dirac Hamiltonian vanishes: $H^{(1)}_{(D)ADM} \approx 0$, and the final 4-metric becomes

$${}^4\hat{g}_{AB}(\tau,\vec{\sigma}) = \epsilon \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{rs} \end{pmatrix}.$$

ix) In void spacetimes the two gauge-fixings $\rho(\tau,\vec{\sigma}) \approx 0$ and ${}^3\hat{K}(\tau,\vec{\sigma}) \approx 0$ are equivalent and one chooses $\phi(\tau,\vec{\sigma}) = 1$ [i.e. $q(\tau,\vec{\sigma}) = 0$]; in this gauge one has ${}^3\hat{R} = 0$ for the 3-hypersurfaces $\Sigma_\tau$ [they have both the scalar curvature and the trace of the extrinsic one vanishing], but in other gauges the 3-curvature and the trace of the extrinsic one may be not vanishing because the solution $\phi(\tau,\vec{\sigma})$ of the reduced Lichnerowicz equation may become nontrivial. From Eqs.(40), the weak and strong Poincaré charges are

$$\begin{aligned}
\hat{P}^\tau_{ADM,R} = &-\epsilon \int d^3\sigma \Big( \frac{\phi^{-6}(\tau,\vec{\sigma})}{24k} [-(\phi^{-4}\rho^2)(\tau,\vec{\sigma}) - \\
&- 2(\phi^{-2}\rho)(\tau,\vec{\sigma})\delta_{r(a)} \int d^3\sigma_1 \sum_n \mathcal{T}^r_{(a)n}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1) + \\
&+ \frac{1}{3}\sum_{rs}(\delta_{rs}\delta_{(a)(b)} + \delta_{r(b)}\delta_{s(a)} - \delta_{r(a)}\delta_{s(b)})
\end{aligned}$$

$$\int d^3\sigma_1 \sum_m \mathcal{T}^r_{(a)m}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)$$

$$\int d^3\sigma_2 \sum_n \mathcal{T}^s_{(b)n}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_2)] -$$

$$- 8k[\phi^2 \sum_r (\partial_r ln\, \phi)^2](\tau,\vec{\sigma})\Big) \rightarrow_{\rho\rightarrow 0} 8\epsilon k \int d^3\sigma [\phi^2 \sum_r (\partial_r ln\, \phi)^2](\tau,\vec{\sigma})$$

$$\rightarrow_{\phi\rightarrow const.} 0,$$

$$\hat{P}^r_{ADM,R} = \frac{2}{3}\int d^3\sigma \phi^{-2}(\tau,\vec{\sigma})\Big(\Big(\rho\phi^{-2}\partial_r ln\, \phi\Big)(\tau,\vec{\sigma}) +$$

$$+ \sum_{uv}\int d^3\sigma_1 \Big[\partial_r ln\, \phi(\tau,\vec{\sigma})\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0] -$$

$$- \partial_u ln\, \phi(\tau,\vec{\sigma})(\delta_{r(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^r_{(a)v})(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0]\Big](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)\Big)$$

$$\rightarrow_{\rho\rightarrow 0} 0,$$

$$\hat{J}^{rs}_{ADM,R} = \frac{1}{3}\int d^3\sigma [\phi^{-4}\rho](\tau,\vec{\sigma})\Big[\sigma_s\partial_r ln\, \phi - \sigma_r\partial_s ln\, \phi\Big](\tau,\vec{\sigma}) +$$

$$+ \frac{1}{3}\sum_{uv}\int d^3\sigma d^3\sigma_1 \phi^{-2}(\tau,\vec{\sigma})$$

$$\Big[(\sigma_s\partial_r ln\, \phi - \sigma_r\partial_s ln\, \phi)(\tau,\vec{\sigma})\delta_{u(a)}\mathcal{T}^u_{(a)v}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0] -$$

$$- \partial_u ln\, \phi(\tau,\vec{\sigma})[\sigma^s(\delta_{r(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^r_{(a)v}) -$$

$$- \sigma^r(\delta_{s(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^s_{(a)v})(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0]\Big](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)$$

$$\rightarrow_{\rho\rightarrow 0} 0,$$

$$\hat{J}^{\tau r}_{ADM,R} = \epsilon\int d^3\sigma \sigma^r\Big(\frac{\phi^{-6}(\tau,\vec{\sigma})}{24k}[-(\phi^{-4}\rho^2)(\tau,\vec{\sigma}) -$$

$$- 2(\phi^{-2}\rho)(\tau,\vec{\sigma})\delta_{r(a)}\int d^3\sigma_1 \sum_n \mathcal{T}^r_{(a)n}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1) +$$

$$+ \frac{1}{3}\sum_{rs}(\delta_{rs}\delta_{(a)(b)} + \delta_{r(b)}\delta_{s(a)} - \delta_{r(a)}\delta_{s(b)})$$

$$\int d^3\sigma_1 \sum_m \mathcal{T}^r_{(a)m}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)$$

$$\int d^3\sigma_2 \sum_n \mathcal{T}^s_{(b)n}(\vec{\sigma},\vec{\sigma}_2,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_2)] -$$

$$- 8k[\phi^2 \sum_r (\partial_r ln\, \phi)^2](\tau,\vec{\sigma})\Big) +$$

$$+ 4\epsilon k\int d^3\sigma \Big[\phi^{-2}(\phi^4 - 1)\partial_r ln\, \phi\Big](\tau,\vec{\sigma})$$

$$\rightarrow_{\rho\rightarrow 0} \epsilon\int d^3\sigma \{\sigma^r[\phi^2 \sum_u (\partial_u ln\, \phi)^2](\tau,\vec{\sigma}) -$$

$$- 2k\sum_u \delta^r_u\Big(\phi^{-2}\sum_s (\phi^4 - 1)(\delta_{us} - 1)\partial_u ln\, \phi\Big)(\tau,\vec{\sigma})\}$$

$$\rightarrow_{\phi\rightarrow 1} 0,$$

$$P^\tau_{ADM,R} = \hat{P}^\tau_{ADM,R} + \int d^3\sigma \hat{\mathcal{H}}_R(\tau,\vec{\sigma}) = -8\epsilon k\sum_u \int_{S^2_{\tau,\infty}} d^2\Sigma_u\, \{\phi\partial_u\phi\}(\tau,\vec{\sigma})$$

$$
\begin{aligned}
&\rightarrow_{\phi\rightarrow const.} 0,\\
P^r_{ADM,R} &= \hat{P}^r_{ADM,R} = -\frac{1}{3}\int_{S^2_{\tau,\infty}} d^2\Sigma_r[\phi^{-2}\rho](\tau,\vec{\sigma}) -\\
&- \frac{1}{6}\sum_{uv}\int_{S^2_{\tau,\infty}} d^2\Sigma_u\,\phi^{-2}(\tau,\vec{\sigma})\int d^3\sigma_1\\
&(\delta_{r(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^r_{(a)v})(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)\\
&\rightarrow_{\rho\rightarrow 0} 0,\\
J^{rs}_{ADM,R} &= \hat{J}^{rs}_{ADM,R} =\\
&= \frac{1}{6}\sum_u\int_{S^2_{\tau,\infty}} d^2\Sigma_u(\sigma^r\delta^s_u - \sigma^s\delta^r_u)(\phi^{-4}\rho)(\tau,\vec{\sigma}) +\\
&+ \frac{1}{12}\sum_{uv}\int_{S^2_{\tau,\infty}} d^2\Sigma_u\int d^3\sigma_1\Big[\sigma^r(\delta_{s(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^s_{(a)v}) -\\
&- \sigma^s(\delta_{r(a)}\mathcal{T}^u_{(a)v} + \delta_{u(a)}\mathcal{T}^r_{(a)v})\Big](\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0](\phi^{-2}\rho)(\tau,\vec{\sigma}_1)\\
&\rightarrow_{\rho\rightarrow 0} 0,\\
J^{\tau r}_{ADM,R} &= \hat{J}^{\tau r}_{ADM,R} + \frac{1}{2}\int d^3\sigma\sigma^r\hat{\mathcal{H}}_R(\tau,\vec{\sigma}) =\\
&= -16\epsilon k\sum_u\int_{S^2_{\tau,\infty}} d^2\Sigma_u\sigma^r(\phi\partial_u\phi)(\tau,\vec{\sigma}) +\\
&+ 2\epsilon k\int_{S^2_{\tau,\infty}} d^2\Sigma_r[\phi^{-2}(\phi^4-1)](\tau,\vec{\sigma})\\
&\rightarrow_{\phi\rightarrow 1} 0.
\end{aligned}
\tag{59}
$$

This shows that at the level of Dirac brackets with respect to the natural gauge fixing $\rho(\tau,\vec{\sigma}) \approx 0$ [i.e. with respect to the pair of second class constraints $\rho \approx 0$, $\phi - 1 \approx 0$] the ten weak and strong Poincaré charges vanish for the solution $\phi(\tau,\vec{\sigma}) = 1$ selected by the boundary conditions (37) [so that they must vanish in all the others gauges connected with this solution, being conserved gauge invariant quantities].

x) Since $r_{\bar{a}}(\tau,\vec{\sigma}) = \pi_{\bar{a}}(\tau,\vec{\sigma}) = 0$ are solutions of the two independent dynamical equations contained in Einstein's equations, void spacetimes are an equivalence class of "Hamiltonian dynamical gravitational fields" and not a kinematical one (see the Conclusions of II and Appendix A). For them the group manifold of the Hamiltonian gauge transformations is restricted and only gauge transformations which are also dynamical symmetries of Einstein's equations (i.e. solutions of the associated Jacobi equations) are allowed. This implies that void spacetimes are described by the standard equivalence class of "Einstein or dynamical gravitational fields" corresponding to the flat 4-geometry, one of whose representative is Minkowski spacetime in rectangular coordinates. This is verified in Appendix E.

As we shall see in Ref. [38], in presence of matter, in the static limit $|r_{\bar{a}}(\tau,\vec{\sigma})| << 1$, $|\pi_{\bar{a}}(\tau,\vec{\sigma})| << 1$, Eq.(58) becomes the Poisson equation $-\triangle_{FLAT}\phi = \rho_{matter} + O(r_{\bar{a}},\pi_{\bar{a}})$, showing that $\phi(\tau,\vec{\sigma})$ is the general relativistic generalization of the Newton potential [see also the term M in the boundary conditions (37)]. But now the Poincaré charges are not vanishing so that we cannot use void spacetimes as approximations of spacetimes $M^4$ for extremely weak gravitational fields. This is contrary to the expectation that for weak gravitational fields a spacetime can be approximated by a void spacetime with "test" matter, but it is consistent with parametrized Minkowski theory for that "test" matter: its arbitrary

spacelike hypersurfaces embedded in Minkowski spacetime describe a family of accelerated observers much bigger of the one allowed in tetrad gravity (namely the WSW hypersurfaces, which do not exist in void spacetimes). The implications of this fact, which have still to be investigated, are that most of the possible accelerated reference systems of Minkowski spacetime (which are the starting point for the classical basis of the Unruh effect) are unrelated with general relativity, at least with its canonical ADM formulation presented in this paper. Also, more study will be needed to clarify the conceptual difference between a "test particle" following a geodesic of an external gravitational field and a "dynamical particle plus the gravitational field" (such a particle will not follow a geodesic of the resulting dynamical gravitational field).

Given the previous interpretation of the conformal factor $\phi$, let us remark that the definition of the so called "gravitoelectric field" as minus the gradient of the Newton potential plus the post-Newtonian gravitoelectric corrections [104] (see also Ref. [48]) [describing effects like gravitational redshift, perihelion precession of Mercury, bending of light and Saphiro's radar time delay], becomes with our notation $\vec{E}_G = -\vec{\partial}\phi(\tau,\vec{\sigma})$: it goes to zero for $\phi \rightarrow 1$ in void spacetimes. Analogously, the "gravitomagnetic vector potential" is $\vec{A}_G(\tau,\vec{\sigma}) = \{{}^4g_{\tau r}(\tau,\vec{\sigma}) = -\epsilon N_r(\tau,\vec{\sigma})\}$, and the "gravitomagnetic field" is $\vec{B}_G(\tau,\vec{\sigma}) = \vec{\partial} \times \vec{A}_G(\tau,\vec{\sigma}) = c\vec{\Omega}(\tau,\vec{\sigma})$ [with $\vec{\Omega}$ the angular velocity connected with the precessional effects like De Sitter and Lense-Thirring effects and the dragging of inertial frames] go to zero for $\vec{\tilde{\lambda}}(\tau) = 0$ and $\rho(\tau,\vec{\sigma}) \approx 0$. The reformulation of gravitomagnetism in ADM tetrad gravity will be the subject of a future paper.

Another use of the terms "gravitoelectric" and "gravitomagnetic" effects is connected with the electric and magnetic parts of the Weyl tensor (see the end of Appendix A of I and Refs. [105]). By choosing the normals to $\Sigma_\tau$ as privileged timelike 4-vectors, one has ${}^4E_{rs} = {}^4C_{r\tau s\tau}$, ${}^4H_{rs} = \frac{1}{2}\epsilon_{s\tau}{}^{uv}\, {}^4C_{r\tau uv}$ (see Appendix A of I), which go to zero for $\phi \rightarrow 1$ and $\rho \rightarrow 0$ (see Appendix E) in void spacetimes. While the electric part represents the tidal force of the curvature, the magnetic part, which has no Newtonian analogue, is generated by the vorticity, shear, .... of the congruence of the timelike worldlines used for the decomposition .

## VII. CONCLUSIONS.

In this paper and in the previous two I and II we studied the canonical reduction of tetrad gravity in absence of matter, we investigated its Hamiltonian group of gauge transformations, we found a canonical basis of Dirac's observables for the Hamiltonian kinematical and dynamical gravitational fields (see the Conclusions of II) and we have solved many interpretational problems at every level of the theory.

i) A modification of ADM tetrad gravity along lines suggested by Dirac has been proposed to give a solution to the deparametrization problem of general relativity so to recover parametrized Minkowski theories restricted to spacelike hyperplanes in the limit of absence of gravity. The requirement of absence of supertranslations restricts the allowed coordinate systems and the boundary conditions of the fields and of the gauge transformations generated by the first class constraints. As a consequence, the Hamiltonian formalism and the Hamiltonian group of gauge transformations turns out to be well defined for the family of spacetimes identified by Christodoulou and Klainermann. The allowed 3+1 splittings of spacetime are only the Wigner-Sen-Witten (WSW) family of foliations, with the leaves tending to Minkowski spacelike hyperplanes in a direction-independent way at spatial infinity, having the asymptotic normal parallel to the weak (timelike) ADM 4-momentum and corresponding to the family of foliations of Minkowski spacetime, when matter is present, with the Wigner hyperplanes intrinsically defined by the isolated system. Therefore, when tetrad gravity with matter is restricted to WSW spacelike hypersurfaces, one gets a generalized "rest-frame Wigner-covariant 1-time instant form" description of the dynamics of a globally hyperbolic, asymptotically flat at spatial infinity, spacetime $M^4$ with spacelike slices $\Sigma_\tau$ diffeomorphic to $R^3$ and with the given matter content (matter will be treated in Ref. [38] starting with scalar particles).

The allowed 3+1 splittings of these spacetimes have the spacelike hypersurfaces tending asymptotically to Minkowski hyperplanes in a direction-independent way and at spatial infinity there are preferred (inertial in the rest frame) observers, which however are not static but dynamically defined. They replace static concepts like the "fixed stars" in the study of the dragging of inertial frames. Since the WSW hypersurfaces and the 3-metric on them are dynamically determined (the solution of Einstein equations is needed to find the physical 3-metric, the allowed WSW hypersurfaces and the Sen connection), one has neither a static background on system-independent hyperplanes like in parametrized Newton theories nor a static one on the system-dependent Wigner hyperplanes like in parametrized Minkowski theories. Now both the WSW hyperplanes and the metric on it are system dependent.

The equivalence class of dynamical "flat" spacetimes (i.e. without gravitational field degrees of freedom), containing Minkowski spacetime in Cartesian coordinates, turns out to be special, because its Poincaré charges vanish, so that there are no WSW hypersurfaces in them. These "void spacetimes" can be defined only in absence of matter (consistently with parametrized Minkowski theories, which exist only in presence of matter) and describe pure acceleration effects without dynamical gravitational field (no tidal effects) allowed in flat spacetimes as the relativistic generalization of Galilean non inertial observers. Therefore, they cannot be used to describe "test matter in flat spacetimes" in some post-Minkowskian approximation. Instead in Ref. [38] we shall study the action-at-a- distance instantaneous effects on scalar particles implied by Einstein theory in the ideal limit of a negligible gravi-

tational field (a more realistic situation with tidal effects will be possible only after having studied the linearization of tetrad gravity in 3-orthogonal gauges) and it will be shown that already this kind of post-Minkowskian approximation lives in non trivial spacetimes (with non trivial WSW hypersurfaces) not gauge equivalent to Minkowski spacetime in Cartesian coordinates.

Moreover, we showed that parametrized Minkowski theories on arbitrary spacelike hypersurfaces are not connected to general relativity since there is not agreement with the general relativistic lapse and shift functions: they seem to be connected with the description of physics in a more general class of accelerated frames, which are not well defined in asymptotically flat at spatial infinity general relativity, so that one will have to re-examine the classical background of the Unruh effect.

The final rest-frame instant form of tetrad gravity on WSW hypersurfaces labelled by the time parameter $\tau \equiv T_{(\infty)}$ in the special 3-orthogonal gauge with $\rho(\tau, \vec{\sigma}) \approx 0$, assuming to know the solution $\phi[r_{\bar{a}}, \pi_{\bar{a}}](\tau, \vec{\sigma})$ of the reduced Lichnerowicz equation (45), is based on the pair of canonical variables $r_{\bar{a}}(\tau, \vec{\sigma})$, $\pi_{\bar{a}}(\tau, \vec{\sigma})$ satisfying the Hamilton equations (the only independent dynamical combinations of Einstein's equations)

$$\begin{aligned} \partial_\tau r_{\bar{a}}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{ r_{\bar{a}}(\tau, \vec{\sigma}), \hat{P}^\tau_{ADM,R} \}, \\ \partial_\tau \pi_{\bar{a}}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{ \pi_{\bar{a}}(\tau, \vec{\sigma}), \hat{P}^\tau_{ADM,R} \}, \end{aligned} \tag{60}$$

where the final Dirac Hamiltonian is the weak ADM energy $\hat{P}^\tau_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi[r_{\bar{a}}, \pi_{\bar{a}}]]$ of Eqs.(53). In these equations there is also the final form of the weak ADM 3-momentum, whose vanishing gives the three first class constraints defining the rest frame

$$\hat{P}^r_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi[r_{\bar{a}}, \pi_{\bar{a}}]] \approx 0, \tag{61}$$

whose natural gauge fixing is [$\hat{J}^{\tau r}_{ADM,R}$ is the weak ADM boost]

$$\sigma^r_{ADM} \approx - \frac{\hat{J}^{\tau r}_{ADM,R}}{\hat{P}^\tau_{ADM,R}} \approx 0. \tag{62}$$

In this gauge the cotriad and the 3-metric have the following form

$$\begin{aligned} {}^3\hat{e}_{(a)r}(\tau, \vec{\sigma}) &= \delta_{(a)r} \Big[ \phi^2[r_{\bar{a}}, \pi_{\bar{a}}] e^{\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \Big](\tau, \vec{\sigma}), \\ {}^3\hat{g}_{rs}(\tau, \vec{\sigma}) &= \delta_{rs} \Big[ \phi^4[r_{\bar{a}}, \pi_{\bar{a}}] e^{\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \Big](\tau, \vec{\sigma}), \end{aligned} \tag{63}$$

while the lapse and shift functions are given by [$N_{(a)} = {}^3\hat{e}^r_{(a)} N_r$]

$$\begin{aligned} N(\tau, \vec{\sigma}) &= -\epsilon + \hat{n}(\tau, \vec{\sigma} | r_{\bar{a}}, \pi_{\bar{a}}, \phi[r_{\bar{a}}, \pi_{\bar{a}}]], \\ N_r(\tau, \vec{\sigma}) &= \hat{n}_r(\tau, \vec{\sigma} | r_{\bar{a}}, \pi_{\bar{a}}, \phi[r_{\bar{a}}, \pi_{\bar{a}}]], \end{aligned} \tag{64}$$

with $\hat{n}$ and $\hat{n}_r$ determined by Eqs.(47) and (39) respectively [in these equations we put $\tilde{\lambda}_\tau(\tau) = \epsilon$, $\tilde{\lambda}_r(\tau) = 0$].

The final form of the 4-metric in coordinates adapted to the WSW hypersurface is

$$
{}^4\hat{g}_{AB} = \epsilon \begin{pmatrix} \Big( -\epsilon + \hat{n} \Big)^2 - \phi^{-4} \sum_r e^{-\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \hat{n}_r^2 & -\hat{n}_s \\ -\hat{n}_r & -\delta_{rs} \phi^4 e^{\frac{2}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \end{pmatrix}. \qquad (65)
$$

ii) Our quasi-Shanmugadhasan canonical transformation, defined in II starting from a multi-time formalism [108], and the choice of 3-orthogonal coordinates transforms the superhamiltonian constraint in a nonlinear integro-differential equation (the reduced Lichnerowicz equation) for the conformal factor $\phi = e^{q/2}$ of the 3-metric, whose conjugate momentum $\pi_\phi = 2\phi^{-1}\rho$ plays the role of the last gauge variable. Since this gauge variable describes nonlocal properties of the extrinsic curvature of the Cauchy surfaces $\Sigma_\tau$, its variation describes the allowed 3+1 splitting of spacetime (the ADM theory is independent from the choice of anyone of them) : this is the meaning of the gauge transformations generated by the superhamiltonian constraint, whose effect is to change $\rho$ and, therefore, the extrinsic curvature of the leaves.

This fact leads to the distinction between Hamiltonian kinematical and dynamical gravitational fields made in the Conclusions of II. As shown there (see also Appendix A) on the solutions of Einstein's equations we must restrict the parameters of the gauge transformations (and, therefore, the gauge variables) in such a way that the allowed gauge transformations are also dynamical symmetries of the Einstein's equations: this fact implies that the Hamiltonian dynamical gravitational fields coincide with the standard "Einstein or dynamical gravitational fields", namely with a single 4-geometry whose 4-metrics are solutions of Einstein's equations (this 4-geometry is parametrized in terms of one conformal 3-geometry).

The knowledge of the solution $\phi(\tau,\vec{\sigma}) = e^{q(\tau,\vec{\sigma})/2} = F[\rho, r_{\bar{a}}, \pi_{\bar{a}}](\tau,\vec{\sigma})$ of the reduced Lichnerowicz equation in the 3-orthogonal gauge would allow to start the search of the final Shanmugadhasan canonical transformation $q, \rho, r_{\bar{a}}, \pi_{\bar{a}} \mapsto q' = q - 2ln\, F, \rho, \tilde{r}_{\bar{a}}, \tilde{\pi}_{\bar{a}}$ which would implement Kuchar's program defined in Refs. [106,107] and would be an alternative to the York map [31], with $\tilde{r}_{\bar{a}}$, $\tilde{\pi}_{\bar{a}}$ describing the independent degrees of freedom of the gravitational field. However, by adding the natural gauge fixing $\rho(\tau,\vec{\sigma}) \approx 0$ and by going to Dirac's brackets [to eliminate $q, \rho$ even if one has not been able to solve explicitly the Lichnerowicz equation], one finds that $r_{\bar{a}}$, $\pi_{\bar{a}}$ are the natural canonical variables of the gravitational field in this gauge.

Regarding other approaches to the observables in general relativity see also Refs. [109]: the "perennials" introduced in this approach are essentially our Dirac observables. See Ref. [110] for the difficulties in observing perennials experimentally at the classical and quantum levels and in their quantization. See also Ref. [111] on the non existence of observables for the vacuum gravitational field in a closed universe, built as spatial integrals of local functions of Cauchy data and their first derivatives.

Our approach violates the geometrical structure of general relativity breaking general covariance (but in a way associated with the privileged presymplectic Darboux bases naturally selected by the Shanmugadhasan canonical transformations) and avoiding the "spacetime problem" with the choice of the privileged WSW foliations, but it allows the deparametrization of general relativity and a soldering with parametrized Minkowski theories (and parametrized Newton theories for $c \rightarrow \infty$) and to make contact with the kinematical framework, which will be used [67] to find the Tomonaga-Schwinger asymptotic states needed for relativistic bound states (the Fock asymptotic states have no control on the relative times of the asymptotic particles). The problem whether general covariance may be

recovered at the quantum level has to be attacked only after having seen if this minimal quantization program can work.

iii) Let us add some comments on time in general relativity in our case of globally hyperbolic asymptotically flat at spatial infinity spacetimes with the modified ADM tetrad theory.

In general relativity, Isham [112] and Kuchar [107] have made a complete review of the problem of time [see Ref. [113] for a recent contribution to the problem], showing that till now there is no consistent quantization procedure for it.

We are in a scheme in which time is identified before quantization. The unphysical mathematical 1-time of our rest-frame instant form of dynamics on WSW hypersurfaces is the rest-frame "global time" $T_{(\infty)} = p_{(\infty)} \cdot \tilde{x}_{(\infty)} / \sqrt{\epsilon p^2_{(\infty)}} = \hat{P}_{ADM} \cdot x_{(\infty)} / \sqrt{\epsilon \hat{P}^2_{ADM}} = \tau$ [let us note that this is possible for globally hyperbolic, asymptotically flat at spatial infinity, spacetimes; instead a global time does not exist, even with a finite number of degrees of freedom, when the configuration space is compact; see for instance Refs. [109]] and not an internal time. It is the gauge-fixing $T_{(\infty)} - \tau \approx 0$ to the extra Dirac constraint $\epsilon_{(\infty)} - \sqrt{\epsilon \hat{P}^2_{ADM}} \approx 0$ which identifies the foliation parameter with the rest-frame time. The evolution in $T_{(\infty)} = \tau$ of the two canonical pairs of gravitational degrees of freedom is governed by the weak ADM energy $\hat{P}^\tau_{ADM}$.

The positions of the non-covariant "external" center-of-mass variable $\tilde{x}^{(\mu)}_{(\infty)}(\tau)$, replacing the arbitrary origin $x^{(\mu)}_{(\infty)}$ of the coordinates on the WSW hypersurfaces, and of this origin are irrelevant, because, as already said, at the end the 6 variables $\vec{z}_{(\infty)}$, $\vec{k}_{(\infty)}$ are decoupled: they describe the "external" center of mass of the isolated universe or equivalently a decoupled external observer with his "point particle clock" [therefore one does not need "matter clocks and reference fluids" [107,114]]. They are not to be quantized because they can be said to belong to the classical part of the Copenhagen interpretation, but their non-covariance is fundamental in defining the classical Möller radius $|\hat{\vec{S}}_{ADM}| / \hat{P}^\tau_{ADM}$ [where, due to $\hat{\vec{P}}_{ADM} \approx 0$, we have $|\hat{\vec{S}}_{ADM}| = \sqrt{-\epsilon W^2_{ADM}} / \hat{P}^\tau_{ADM}$ with $W^A_{ADM}$ the asymptotic Pauli-Lubanski 4-vector] to be used as a ultraviolet cutoff.

The "internal" center-of-mass 3-variable $\vec{\sigma}_{ADM}[r_{\bar{a}}, \pi_{\bar{a}}]$ (built in terms of the weak Poincaré charges as it is done for the Klein-Gordon field on the Wigner hyperplane in Ref. [59]; due to $\hat{P}^r_{ADM} \approx 0$ we have $\sigma^r_{ADM} \approx -\hat{J}^{\tau r}_{ADM} / \hat{P}^\tau_{ADM}$) of the universe inside a WSW hypersurface identifies the 3 gauge-fixings $\vec{\sigma}_{ADM} \approx 0$ to be added to $\hat{\vec{P}}_{ADM}[r_{\bar{a}}, \pi_{\bar{a}}] \approx 0$. With these gauge fixings this point coincides with the arbitrary origin $x^{(\mu)}_{(\infty)}(\tau)$. With $\vec{\sigma}_{ADM} \approx 0$ the origin $x^{(\mu)}_{(\infty)}(\tau)$ becomes simultaneously [59] the Fokker-Price center of inertia, the Dixon center of mass and Pirani and Tulczjyew centroids of the universe, while the non-covariant "external" center-of-mass variable $\tilde{x}^{(\mu)}_{(\infty)}(\tau)$ is the analog of the Newton-Wigner position operator.

Our final picture of the reduced phase space has similarities the frozen Jacobi picture of Barbour [115] and his proposal to substitute time with the astronomical ephemeris time [116] [in his timeless and frameless approach based on Ref. [117] the local ephemeris time coincides with the local proper time] may be a starting point for correlating local physical clocks with the mathematical time parameter $\tau = T_{(\infty)}$ of the foliation [and not for defining a timeless theory by using Jacobi's principle]. We think that scenario a) of Section III, used for the description of void spacetimes without matter, is a realization of the fully Machian

approach of Barbour which, however, seems possible only in absence of matter. Instead the scenario b) with a decoupled free "external" center-of-mass variable is Machian only in the fact that there are only dynamical relative variables left both in asymptotically flat general relativity and in parametrized Minkowski theories.

Let us remark that the interpretation of the superhamiltonian constraint as a generator of gauge transformations with natural gauge-fixing $\rho(\tau, \vec{\sigma}) \approx 0$ (at least in 3-orthogonal coordinates) leads to the conclusion that neither York's internal extrinsic time nor Misner's internal intrinsic time are to be used as time parameters: Misner's time (the conformal factor) is determined by the Lichnerowicz equation while York's time (the trace of the extrinsic curvature) by the natural gauge-fixing $\rho \approx 0$. This implies a refusal of the standard (interconnected) interpretations of the superhamiltonian constraint either as a generator of time evolution (being a time-dependent Hamiltonian) like in the commonly accepted viewpoint based on the Klein-Gordon interpretation of the quantized superhamiltonian constraint, i.e. the Wheeler-DeWitt equation [see Kuchar in Ref. [118], Wheeler's evolution of 3-geometries in superspace in Ref. [117,119] and the associated "sandwich conjecture" (see for instance Refs. [120], [117] and the review in Ref. [107]); see Ref. [121] for the cosmological implications] or as a quantum Hamilton-Jacobi equation without any time [one can introduce a concept of evolution, somehow connected with an effective time, only in a WKB sense [122]].

Since the superhamiltonian constraint is quadratic in the momenta, one is naturally driven to make a comparison with the free scalar relativistic particle with the first class constraint $p^2 - \epsilon m^2 \approx 0$. As shown in Refs. [10,123], the constraint manifold in phase space has 1-dimensional gauge orbits (the two disjointed branches of the mass-hyperboloid); the $\tau$-evolution generated by the Dirac Hamiltonian $H_D = \lambda(\tau)(p^2 - \epsilon m^2)$ gives the parametrized solution $x^\mu(\tau)$. Instead, if we go to the reduced phase space by adding the non-covariant gauge-fixing $x^o - \tau \approx 0$ and eliminating the pair of canonical variables $x^o \approx \tau$, $p^o \approx \pm\sqrt{\vec{p}^2 + m^2}$, we get a frozen Jacobi description in terms of independent Cauchy data, in which the same Minkowski trajectory of the particle can be recovered in the non-covariant form $\vec{x}(x^o)$ by introducing as Hamiltonian the energy generator $\pm\sqrt{\vec{p}^2 + m^2}$ of the Poincaré group [with the variables of Ref. [124], one adds the covariant gauge-fixing $p \cdot x/\sqrt{p^2} - \tau \approx 0$ and eliminates the pair $T = p \cdot x/\sqrt{p^2}$, $\epsilon = \eta\sqrt{p^2} \approx \pm m$; now, since the invariant mass is constant, $\pm m$, the non-covariant Jacobi data $\vec{z} = \epsilon(\vec{x} - \vec{p}x^o/p^o)$, $\vec{k} = \vec{p}/\epsilon$ cannot be made to evolve]. Instead the superhamiltonian constraint, being a secondary first class constraint of a field theory, has an associated "Gauss law" (see Eq.(A5) of Appendix A) like the supermomentum constraints, even if it is quadratic in the momenta and this fact is connected to the definition of the ADM energy. This Gauss law (defining the strong ADM energy as a surface integral) shows that the superhamiltonian constraint has to be solved in the conformal factor q or in $\phi = e^{q/2}$. Therefore, its gauge orbit in superspace is parametrized by $\rho$ and this is not a time evolution (instead it is connected with the normal deformations of the spacelike hypersurfaces so to change a 3+1 splitting in another one): the solution of the superhamiltonian constraint do not define the weak ADM energy, which, instead, is connected with an integral over 3-space of that part of the superhamiltonian constraint dictated by the Gauss law. This does not contradict Teitelboim's results [125]: the algebra of supermomentum and superhamiltonian constraints reflects the embeddability of $\Sigma_\tau$ into $M^4$ (see also Ref. [56]): the theory is simply independent from the allowed 3+1 splittings with embeddable leaves $\Sigma_\tau$.

Let us remember that in Ref. [126] the nonrelativistic limit of the ADM action for metric gravity was considered: it allowed the identification of a singular Lagrangian density with general Galileo covariance depending on 27 fields [coming from the development in series of powers of $1/c^2$ of $N$, $N_r$, ${}^3g_{rs}$] describing Newton gravity in arbitrary coordinates. This theory has first class constraints connected with inertial forces and second class constraints, determining the static Newton potential in arbitrary frames of reference when massive particles are present [see Ref. [127] for alternative nonrelativistic gravity theories]. This implies that it will be possible to consider the nonrelativistic limit of our modified tetrad gravity and establish its connections with the postNewtonian approximations [128], in particular the recent one of Ref. [129,130]: see Appendix G for some preliminary comments.

Now, at the nonrelativistic level there is an absolute time t and the evolution in this numerical parameter of every system is described by the Hamilton equations associated with a Hamiltonian function H describing the energy of the system [it is a generator of the kinematical (extended) Galileo group when the system is isolated]. Alternatively, one can use a parametrized reformulation of the system by enlarging phase space with a canonical pair $t$, $E$ [$\{t, E\} = \epsilon = \pm 1$, if $\epsilon$ is the signature of the time axis], by adding the first class constraint $\chi = E - H \approx 0$ (so that the Dirac Hamiltonian is $H_D = \lambda(\tau)\chi$) and by calling $\tau$ the scalar parameter associated with the canonical transformations generated by $\chi$. The parameter $\tau$ labels the leaves of a foliation of Galilei spacetime; the leaves are (rest-frame) hyperplanes, which are the limit of Wigner hyperplanes in parametrized Minkowski theories for $c \rightarrow \infty$. One gets a parametric description of the same physics with t and the solutions of the original Hamilton equations now expressed as functions of the new time parameter $\tau$. If one adds the gauge-fixing $t - \tau \approx 0$ , one gets a frozen reduced phase space (equal to the original one) like in the Jacobi theory, in which one reintroduce an evolution by using the energy E=H for the evolution in $t = \tau$. However, with more general gauge-fixings $t - f(\tau, ...) \approx 0$, where dots mean other canonical variables, the associated Hamiltonian is no more the energy [see Ref. [123]].

In the standard nonrelativistic quantization of the system one defines a Hilbert space and writes a Schroedinger equation in which t is a parameter and in which the t-evolution is governed by an operator obtained by quantizing the Hamiltonian function corresponding to the energy [see Ref. [132] for a discussion of this point and of the associated ambiguities and problems]. Instead, in the parametrized theory, one should quantize also the pair $t, E$ (one introduces a unphysical Hilbert space in which the t-dependence of wave functions is restricted to be square integrable) and write a Schroedinger equation in $\tau$ with the quantum Dirac Hamiltonian [see Ref. [108] on this point and on the problem of the unphysical and physical scalar products] and then impose the constraint to identify the physical states. This procedure is ambiguous, because in this way the energy operator has no lower bound for its spectrum in the unphysical Hilbert space and it is delicate to recover the physical Hilbert space from the quotient of the unphysical one with respect to the quantum unitary gauge transformations generated by the quantum constraint. In particular, physical states have infinite unphysical norm [usually the zero eigenvalue belongs to the continuum spectrum of the constraint operators] and the construction of the physical scalar product for physical states (without any restriction on the t-dependence) depends on the form of the constraint (see Ref. [124] for a relativistic example).

Moreover, the absolute time t, which labels the Euclidean leaves of the absolute foliation

of Galileo spacetime, is unrelated to physical clocks. As shown in Ref. [133] (see also Ref. [112]), in the physical Hilbert space there is no operator such that: i) it can be used as a "perfect clock", in the sense that, for some initial state, its observed values increase monotonically with t; ii) is canonically conjugate to the Hamiltonian operator (this would imply that this operator is not definite positive). All this is also related with Rovelli's proposal [134] of replacing t (in the nonrelativistic case) with an "evolving constant of the motion", i.e. a t-dependent function of operators commuting with the Hamiltonian. This proposal can be done either in the standard or in the parametrized version of the theory (see also Ref. [135]); among others [136], Kuchar [107] critics it for the ambiguities coming from the operator ordering problem. In any case there are all the previously mentioned problems and also the fact that the conjugate variables of these evolving constants of motion generically have nothing to do with the energy and can have spectra and symmetries of every type (see Ref. [132]).

All the proposals of replacing the parameter t with some physical time function (or operator) show that this is the main unsolved problem: how to identify (at least locally, possibly globally) the leaves of the foliation of Galileo spacetime with "physical clocks", i.e. with an apparatus described in the given either phase or Hilbert space. See again in this connection Barbour [115] who uses as local time functions special space coordinates (the astronomical ephemeris time [116] or some its relativistic extension).

Now, in the approach based on parametrized special relativistic theories in Minkowski spacetime, the final result is that every isolated system (or better all its configurations with a timelike total 4-momentum) identifies a Wigner foliation of Minkowski spacetime. Its leaves (the Wigner hyperplanes) are labelled by a scalar parameter $T_s = \tau$ (the center-of-mass time in the rest frame) in the "rest-frame Wigner-covariant 1-time instant form" with the evolution in this parameter governed by the invariant mass of the system. There is also a decoupled non-covariant center-of-mass point with free motion. The quantization of this instant form produces a 1-time Schroedinger equation as in the standard unparametrized nonrelativistic case with the Newtonian time t replaced by the Lorentz-scalar rest frame time $T_s$.

In our modified tetrad gravity we have again the same picture in the generalized rest-frame instant form with WSW foliations. Therefore, in our unified approach to general relativity, special relativity and Newton-Galileo theories we are never going to quantize any time variable and the problem of time is replaced by the problem of how to correlate (locally) physical clocks with the mathematical time parameter labelling the leaves of the 3+1 splitting of spacetime.

iv) Let us remark that our ADM tetrad formulation assumed the existence of a mathematical abstract 4-manifold, the spacetime $M^4$, to which we added 3+1 splittings with spacelike leaves $\Sigma_\tau \approx R^3$. The mathematical points of $M^4$ have no physical meaning [invariance under $Diff\, M^4$, hole argument [137]; see the Conclusions of II] and are coordinatized with $\Sigma_\tau$-adapted coordinates $(\tau, \vec{\sigma})$. All fields (also matter fields when present) depend on these mathematical coordinates for $M^4$, but till now there is no justification for saying that the points (or events) of the spacetime have any physical meaning [instead in special relativity they are physical points by hypothesis].

Is it possible to label the points of $M^4$ in terms of Dirac's observables a posteriori by introducing "physical points"? As said in the Conclusions of II, once all gauge freedoms have

been eliminated this can be done, in analogy to what happens with the vector potential of electromagnetism which becomes measurable in a completely fixed gauge like the Coulomb one. Indeed, one can build a system of coordinates for a spacetime $M^4$ without Killing vectors (at least in absence of matter) in terms of the Dirac observables $r_{\bar{a}}$, $\pi_{\bar{a}}$, describing the gravitational field in our complete 3-orthogonal gauge with the natural gauge fixing $\rho(\tau, \vec{\sigma}) \approx 0$ by restricting to that gauge the 4 independent Komar-Bergmann "individuating fields" [3,138] (see also Ref. [117,112]). These fields are bilinears and trilinears in the Weyl tensor (independent from the lapse and shift functions) invariant under $Diff\, M^4$ but not under the Hamiltonian group of gauge transformations. Using the results of Appendices D,E, of II plus the natural gauge fixing the individuating fields can be expressed in terms of Dirac's observables and there is a local 1-1 correspondence between them and the mathematical coordinates $(\tau, \vec{\sigma})$.

v) In Appendix F there is a definition of null tetrads which is natural from the Hamiltonian point of view. It will be used in a future paper to find the phase space expressions of the Newman-Penrose 20 scalars carrying all the information on the Riemann tensor (see for instance Ref. [131]; they are ${}^4R$, nine scalars for the trace-free Ricci tensor and five complex scalars for the Weyl tensor). This will allow to find the dependence of these scalars on the gauge variables of tetrad gravity and to study their asymptotic expansions at null infinity. Moreover, the two spacelike vector fields in the null tetrad will identify in the tangent space in each point of $M^4$ a 2-plane: this will allow to connect our canonical variables $r_{\bar{a}}$, $\pi_{\bar{a}}$ in the special 3-orthogonal gauge with the symmetric trace-free 2-tensors on 2-planes of Ref. [7]. The 2-planes identified in the tangent space in each point of $M^4$ are orthogonal to the time-like normal $l^A(\tau, \vec{\sigma})$ at $\Sigma_\tau$ in that point and to the spacelike unit vector $\mathcal{N}^A(\tau, \vec{\sigma})$, defined in Appendix F (the gauge direction identified by the shift functions), in that point. In general (for instance in the 3-orthogonal gauges) this vector field is not surface-forming, namely the associated differential form $\mathcal{N}_A(\tau, \vec{\sigma}) d\sigma^A$ is not proportional to a closed differential 1-form (non zero vorticity). As said in Appendix F, it will be important to study those coordinate systems for $\Sigma_\tau$ implying that $\mathcal{N}^A(\tau, \vec{\sigma})$ is surface forming, because in them there is the possibility of making a 2+2 decomposition of $M^4$ with a conformal 2-structure [140,141], of having variables in the spirit of the Newman-Penrose formalism and of finding canonical variables $r^{'}_{\bar{a}}$ for the gravitational field naturally defined on 2-surfaces. Our approach opens the path to a systematic search of these 3-coordinates for $\Sigma_\tau$, which should be investigated in the future like the normal coordinates around a point of $M^4$.

vi) Let us now make some comments on the quantization of tetrad gravity in this scheme in which general covariance is completely broken having completely fixed all the gauges. See Ref. [110] for an updated discussion of quantization problems in canonical gravity (and Ref. [118] for the quantization of parametrized theories).

The quantization of the rest-frame instant form of tetrad gravity in the 3-orthogonal gauge with the natural gauge fixing $\rho(\tau, \vec{\sigma}) \approx 0$ by using the mathematical time parameter $T_{(\infty)} \equiv \tau$ (the rest-frame time of the "external" decoupled point particle clock) on the Wigner-Sen-Witten hypersurfaces should be done with the following steps:

a) Assume to have found either the exact or an approximate solution of the classical reduced Lichnerowicz equation $\phi = \phi(r_{\bar{a}}, \pi_{\bar{a}})$ and to have evaluated the associated weak ADM 4-momentum $\hat{P}^A_{ADM,R} = \hat{P}^A_{ADM,R}[r_{\bar{a}}, \pi_{\bar{a}}, \phi(r_{\bar{a}}, \pi_{\bar{a}})]$.

b) On each WSW hypersurface $\Sigma_\tau \approx R^3$ replace the Hamiltonian gravitational field physical

degrees of freedom $r_{\bar a}(\tau,\vec\sigma)$, $\pi_{\bar a}(\tau,\vec\sigma)$ with operators $\hat r_{\bar a}(\tau,\vec\sigma) = r_{\bar a}(\tau,\vec\sigma)$, $\hat\pi_{\bar a}(\tau,\vec\sigma) = i\frac{\delta}{\delta r_{\bar a}(\tau,\vec\sigma)}$ (Schroedinger representation) on some Hilbert space.
c) Write the functional Schroedinger wave equation

$$i\frac{\partial}{\partial\tau}\Psi(\tau,\vec\sigma|r_{\bar a}] = \hat P^{(op)\,\tau}_{ADM,R}[r_{\bar a},\hat\pi_{\bar a},\phi(r_{\bar a},\hat\pi_{\bar a})]\Psi(\tau,\vec\sigma|r_{\bar a}], \tag{66}$$

plus the 3 conditions defining the rest frame

$$\hat P^{(op)\,r}_{ADM,R}[r_{\bar a},\hat\pi_{\bar a},\phi(r_{\bar a},\hat\pi_{\bar a})]\Psi(\tau,\vec\sigma|r_{\bar a}] = 0, \tag{67}$$

after having choosen (if possible!) an ordering such that $[\hat P^{(op)\,A}_{ADM,R},\hat P^{(op)\,B}_{ADM,R}] = 0$. Let us remark that at this stage it could be useful the suggestion of Ref. [139] that the unphysical space of these functionals does not need to be a Hilbert space and that, in it, the observables need not to be self-adjoint operators (these properties must hold only in the physical space with the physical scalar product). This Schroedinger equation has not an“internal Schroedinger interpretation” since neither “Misner internal intrinsic time” nor “York internal extrinsic time” nor any function like the “Komar-Bergmann individuating fields” are the time: it does not use the superhamiltonian constraint (like the Wheeler-DeWitt equation) but the derived weak ADM energy.

The scalar product associated with this Schroedinger equation defines the Hilbert space and the operators $\hat P^{(op)A}_{ADM,R}$ should be self-adjoint with respect to it. Since there are the 3 conditions coming from the 3 first class constraints defining the rest frame, the physical Hilbert space of the wave functionals $\Psi_{phys}$ solution of Eq.(67) will have an induced physical scalar product which depends on the functional form of the constraints $\hat P^r_{ADM,R}\approx 0$ as it can be shown explicitly in finite-dimensional examples [142,124], so that it is not given by a system-independent rule.

Another possibility is to add and quantize also the gauge fixings $\vec\sigma_{ADM}\approx 0$. In this case one could impose the second class constraints in the form $<\Psi|\sigma^{(op)r}_{ADM}|\Psi>=0$, $<\Psi|\hat P^{(op)r}_{ADM,R}|\Psi>=0$ and look whether it is possible to define a Gupta-Bleuler procedure.

The best would be to be able to find the canonical transformation $r_{\bar a}(\tau,\vec\sigma)$, $\pi_{\bar a}(\tau,\vec\sigma)\mapsto \vec\sigma_{ADM}$, $\hat{\vec P}_{ADM,R}$, $R_{\bar a}(\tau,\vec\sigma)$, $\Pi_{\bar a}(\tau,\vec\sigma)$ [$R_{\bar a}$, $\Pi_{\bar a}$ being relative variables], since in this case we would quantize only the final relative variables:

$$\begin{aligned}
\Psi_{phys} &= \tilde\Psi(\tau,\vec\sigma|R_{\bar a}],\\
i\frac{\partial}{\partial\tau}\Psi_{phys} &= \hat E^{(op)}_{ADM}[R_{\bar a},\hat\Pi_{\bar a} = i\frac{\delta}{\delta R_{\bar a}}]\Psi_{phys},\\
&\qquad with\quad \hat E_{ADM} = \hat P^\tau_{ADM,R}[r_{\bar a},\pi_{\bar a},\phi(r_{\bar a},\pi_{\bar a})]|_{\vec\sigma_{ADM}=\hat{\vec P}_{ADM,R}=0}.
\end{aligned} \tag{68}$$

Let us remark that many aspects of the problem of time in quantum gravity [107] would be avoided: i) there would be no “multiple choice problem” since there is only one mathematical time variable $T_{(\infty)}=\tau$; ii) the problem of “functional evolution” would be reduced to find an ordering such that $[\hat P^{(op)\,A}_{ADM,R},\hat P^{(op)\,B}_{ADM,R}]=0$ ; iii) the “Hilbert space problem” is not there because we do not have the Wheeler-DeWitt equation but an ordinary Schroedinger equation; iv) there is a physical ultraviolet cutoff (the Möller radius) like in parametrized Minkowski theories which could help in regularization problems.

Naturally general covariance is completely broken and everything is defined only on the Wigner-Sen-Witten foliation associated with the natural gauge fixing $\rho(\tau, \vec{\sigma}) \approx 0$. If we would do the same quantization procedure in 3-normal coordinates on their WSW hypersurfaces associated with the corresponding natural gauge fixing $\rho_{normal}(\tau, \vec{\sigma}) \approx 0$, we would get a different physical Hilbert space whose being unitarily equivalent to the one in 3-orthogonal coordinates is a completely open problem.

If this quantization can be done, the completely gauge-fixed 4-metric ${}^4g_{AB}$ on the mathematical manifold $M^4$ would become an operator ${}^4\hat{g}_{AB}(\tau, \vec{\sigma}|R_{\bar{a}}, \hat{\pi}_{\bar{a}}]$ with the implication of a quantization of the Dirac observables associated with 3-volumes (the volume element Dirac observable is the solution $\phi$ of the reduced Lichnerowicz equation for $\rho = 0$), 2-areas and lengths. Let us remark that these quantities would not a priori commute among themselves: already at the classical level there is no reason that they should have vanishing Dirac brackets (however, two quantities with compact disjoint supports relatively spacelike would have vanishing Dirac brackets).

Since in the Dirac-Bergmann canonical reduction of tetrad gravity spin networks do not show up (but they could be hidden in the non-tensorial character of the Dirac observables $r_{\bar{a}}$, $\pi_{\bar{a}}$ still to be explored), it is not clear which could be the overlap with Ashtekar-Rovelli-Smolin program [143], which is generally covariant but only after having fixed the lapse and shift functions (so that it is not clear how one can rebuild the spacetime from the 3-geometries) and replaces local variables of the type $r_{\bar{a}}(\tau, \vec{\sigma})$ with global holonimies of the 3-spin connection over closed 3-loops.

If the quantization can be made meaningful, the quantum Komar-Bergmann individuating fields would lead to a quantization of the "physical coordinates" for the spacetime $M^4$. This will give a quantum spacetime connected with non commutative geometry approaches.

Let us also remark that instead of using a solution of the classical reduced Lichnerowicz equation with $\rho(\tau, \vec{\sigma}) = 0$, one could use weak ADM 4-momentum $\hat{P}^{(op)A}_{ADM,R}[r_{\bar{a}}, \hat{\pi}_{\bar{a}}, \phi^{(op)}]$ with $\phi^{(op)}$ an operatorial solution of a quantum operatorial reduced Lichnerowicz equation (not a quantum constraint on the states but the quantization of the classical Lichnerowicz equation with $\rho = 0$ after having gone to Dirac brackets).

Finally, let us observe that even if our approach is more complicated than Ashtekar and string ones, it opens the possibility of a unified description of the four interactions after having learned how to couple the standard SU(3)xSU(2)xU(1) model to tetrad gravity and how to make the canonical reduction of the complete theory. There is a unification of the mathematical tools, namely one needs to learn the properties of special functions like the Wilson lines along geodesics of either su(3)-valued connections or so(3)-valued 3-spin-connections. Moreover, the problem of which choice to make for the function space of the fields associated with the four interactions will require to understand whether the Gribov ambiguity is only a mathematical obstruction to be avoided (in tetrad gravity this would eliminate the isometries and in Yang-Mills theory the gauge symmetries and the gauge copies) or whether there is some physics in it (in this case one should learn how to make the canonical reduction in presence of gauge symmetries, gauge copies and isometries).

Even if it is too early to understand whether our approach can be useful either from a computational point of view (like numerical gravity) or for the search of exact solutions, we felt the necessity to revisit the Hamiltonian formulation of tetrad gravity with its intrinsic naturalness for the search of the physical degrees of freedom of any gauge theory and for

the formulation of quantization rules so that one can have a clear idea of the meaning of the gauge fixings and the possibility to have an insight on the role of the gauge degrees of freedom in the realm of exact solutions where traditionally one starts with suitable parametrizations of the line element $ds^2$ and then uses symmetries to simplify the mathematics.

But before attacking these problems, we have to study tetrad gravity in 3-normal coordinates, its linearization to make contact with the theory of gravitational waves, its coupling to matter and how to define the analog of the post-Minkowskian approximation (formal expansion in series of powers of G; see for instance Ref. [144,129]) starting from the rest-frame instant form on WSW hypersurfaces. Instead in Appendix G there is a comparison of the rest-frame instant form of tetrad gravity with the formulation of Ref. [129,130] of the post-Newtonian approximation.

## APPENDIX A: THE SECOND NOETHER THEOREM FOR ADM METRIC GRAVITY.

In Section V of I there was a review of ADM canonical metric gravity, whose secondary first class constraints and Dirac Hamiltonian are [the $\lambda(\tau,\vec{\sigma})$'s are arbitrary Dirac multipliers; $k = c^3/16\pi G$]

$$\begin{aligned}
\tilde{\mathcal{H}}(\tau,\vec{\sigma}) &= \epsilon[k\sqrt{\gamma}\,{}^3R - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}\,{}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}^{uv}](\tau,\vec{\sigma}) = \\
&= \epsilon[\sqrt{\gamma}\,{}^3R - \frac{1}{k\sqrt{\gamma}}({}^3\tilde{\Pi}^{rs}\,{}^3\tilde{\Pi}_{rs} - \frac{1}{2}({}^3\tilde{\Pi})^2)](\tau,\vec{\sigma}) = \\
&= \epsilon k\{\sqrt{\gamma}[{}^3R - ({}^3K_{rs}\,{}^3K^{rs} - ({}^3K)^2)](\tau,\vec{\sigma}) \approx 0, \\
{}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) &= -2\,{}^3\tilde{\Pi}^{rs}{}_{|s}(\tau,\vec{\sigma}) = -2[\partial_s\,{}^3\tilde{\Pi}^{rs} + {}^3\Gamma^r_{su}\,{}^3\tilde{\Pi}^{su}](\tau,\vec{\sigma}) = \\
&= -2\epsilon k\{\partial_s[\sqrt{\gamma}({}^3K^{rs} - {}^3g^{rs}\,{}^3K)] + \\
&+ {}^3\Gamma^r_{su}\sqrt{\gamma}({}^3K^{su} - {}^3g^{su}\,{}^3K)\}(\tau,\vec{\sigma}) \approx 0,
\end{aligned}$$

$$\begin{aligned}
\{{}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}), {}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}^{'})\} &= {}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}^{'})\,\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^s} + {}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r}, \\
\{\tilde{\mathcal{H}}(\tau,\vec{\sigma}), {}^3\tilde{\mathcal{H}}_r(\tau,\vec{\sigma}^{'})\} &= \tilde{\mathcal{H}}(\tau,\vec{\sigma})\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r}, \\
\{\tilde{\mathcal{H}}(\tau,\vec{\sigma}), \tilde{\mathcal{H}}(\tau,\vec{\sigma}^{'})\} &= [{}^3g^{rs}(\tau,\vec{\sigma}){}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}) + \\
&+ {}^3g^{rs}(\tau,\vec{\sigma}^{'}){}^3\tilde{\mathcal{H}}_s(\tau,\vec{\sigma}^{'})]\frac{\partial\delta^3(\vec{\sigma},\vec{\sigma}^{'})}{\partial\sigma^r},
\end{aligned} \tag{A1}$$

$$\begin{aligned}
H_{(D)ADM} &= H_{(c)ADM} + \int d^3\sigma\,[\lambda_N\,\tilde{\Pi}^N + \lambda^{\vec{N}}_r\,\tilde{\Pi}^r_{\vec{N}}](\tau,\vec{\sigma}), \\
H_{(c)ADM} &= \int d^3\sigma[N\,\tilde{\mathcal{H}} + N_r\,{}^3\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) \approx 0.
\end{aligned} \tag{A2}$$

In Eqs.(25), (26) of I it was shown that the ADM action differs from the Hilbert action by a surface term $\Sigma_{ADM} = -\frac{\epsilon c^3}{8\pi G}\int d^4x\partial_\mu[\sqrt{{}^4g}({}^3Kl^\mu + {}^3a^\mu)]$ (${}^3a^\mu = l^\beta l^\mu{}_{;\beta}$ is the acceleration of the observers travelling along the normal $l^\mu$ at $\Sigma_\tau$), which becomes $-\epsilon\frac{c^3}{8\pi G}\int_{\partial U} d^3\Sigma\,[N\sqrt{\gamma}\,{}^3K] = -\epsilon\frac{c^3}{8\pi G}\int_{\tau_i}^{\tau_f} d\tau N_{(as)}(\tau)\int_{S^2_{\tau,\infty}} d^2\Sigma\sqrt{\gamma}\,{}^2K$ for suitable boundary conditions. In this case, following Ref. [42], we assume that, given a subset $U \subset M^4$ of spacetime, $\partial U$ consists of two slices, $\Sigma_{\tau_i}$ (the initial one) and $\Sigma_{\tau_f}$ (the final one) with outer normals $-l^\mu(\tau_i,\vec{\sigma})$ and $l^\mu(\tau_f,\vec{\sigma})$ respectively, and of a surface $S_\infty$ near space infinity, with outer unit (spacelike) normal $n^\mu(\tau,\vec{\sigma})$, orthogonal to the slices [so that the normal $l^\mu(\tau,\vec{\sigma})$ to every slice is asymptotically tangent to $S_\infty$]. The 3-surface $S_\infty$ is foliated by a family of 2-surfaces $S^2_{\tau,\infty}$ coming from its intersection with the slices $\Sigma_\tau$ [therefore, asymptotically $l^\mu(\tau,\vec{\sigma})$ is normal to the corresponding $S^2_{\tau,\infty}$]. The vector $b^\mu_\tau = z^\mu_\tau = Nl^\mu + N^r b^\mu_r$ is not in general tangent to $S_\infty$. In Ref. [42] it is shown that, if the lapse function tends to an asymptotic limit $N_{(as)}(\tau)$, one gets the trace ${}^2K$ of the 2-dimensional extrinsic curvature of the 2-surface $S^2_{\tau,\infty} = S_\infty \cap \Sigma_\tau$, and that for $N_{(as)}(\tau) = 1$, this surface term coincide with the ADM energy in asymptotically flat spacetimes.

In Eq.(77) of I [see also Ref. [51,52,42]] it was shown that in the Legendre transformation from the ADM Lagrangian to the ADM canonical Hamiltonian there was a second surface

term $2\int_{\partial S} d^2\Sigma_s[N_r\, {}^3\tilde{\Pi}^{rs}](\tau,\vec{\sigma}) = 2\int_{S^2_{\tau,\infty}} d^2\Sigma[N_r\, {}^3\tilde{\Pi}^{rs} n_s](\tau,\vec{\sigma})$ $[n_r = \hat{b}^\mu_r n_\mu]$. Again it was shown in Ref. [42] that, for constant asymptotic shifts, this surface term reproduces the ADM momentum in asymptotically flat spacetimes.

In Refs, [5,6], following Refs. [51,52], it is shown that the differentiability of the ADM canonical Hamiltonian $H_{(c)ADM}$ requires the introduction of the following surface term $H_{(c)ADM} \mapsto H_{(c)ADM} + H_\infty]$

$$H_\infty = -\int_{S^2_{\tau,\infty}} d^2\Sigma_u \quad \{\epsilon k\sqrt{\gamma}\, {}^3g^{uv}\, {}^3g^{rs}[N(\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs}) + \\ + \partial_u N({}^3g_{rs} - \delta_{rs}) - \partial_r N({}^3g_{sv} - \delta_{sv})] - 2N_r\, {}^3\tilde{\Pi}^{ru}\}(\tau,\vec{\sigma}). \qquad (A3)$$

Its use in Section III is connected with the ADM Poincaré generators, so that it is equivalent to the two surfaces terms just discussed.

Let us show how this term arises from a suitable splitting of the superhamiltonian and supermomentum constraints. By using ${}^3\Gamma^s_{sr} = \frac{1}{\sqrt{\gamma}}\partial_r\sqrt{\gamma}$, ${}^3g^{uv}\, {}^3\Gamma^r_{uv} = -\frac{1}{\sqrt{\gamma}}\partial_s(\sqrt{\gamma}\, {}^3g^{rs})$, ${}^3R = {}^3g^{uv}({}^3\Gamma^r_{us}\, {}^3\Gamma^s_{vr} - {}^3\Gamma^r_{uv}\, {}^3\Gamma^s_{sr}) + \frac{1}{\sqrt{\gamma}}\partial_r[\sqrt{\gamma}({}^3g^{uv}\, {}^3\Gamma^r_{uv} - {}^3g^{ur}\, {}^3\Gamma^v_{vu})]$, and ${}^3g^{rs}\, {}^3\Gamma^u_{rs} - {}^3g^{uv}\, {}^3\Gamma^s_{sv} = {}^3g^{rs}\, {}^3g^{uv}(\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs})$, one gets

$$\int d^3\sigma[N\tilde{\mathcal{H}} + N_r\, {}^3\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) = \\ = \int d^3\sigma\{\epsilon k N\partial_u[\sqrt{\gamma}({}^3g^{rs}\, {}^3\Gamma^u_{rs} - {}^3g^{uv}\, {}^3\Gamma^s_{sv})] - 2N_r\partial_u\, {}^3\tilde{\Pi}^{ru}\}(\tau,\vec{\sigma}) + \\ + \int d^3\sigma\{\epsilon k N[\sqrt{\gamma}\, {}^3g^{rs}({}^3\Gamma^u_{rv}\, {}^3\Gamma^v_{su} - {}^3\Gamma^u_{rs}\, {}^3\Gamma^v_{vu}) - \\ - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}\, {}^3\tilde{\Pi}^{rs}\, {}^3\tilde{\Pi}^{uv}] - 2N_r\, {}^3\Gamma^r_{su}\, {}^3\tilde{\Pi}^{su}\}(\tau,\vec{\sigma}) = \\ = \int_{S^2_{\tau,\infty}} d^2\Sigma_u\{\epsilon k N\sqrt{\gamma}\, {}^3g^{rs}\, {}^3g^{uv}(\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs}) - 2N_r\, {}^3\tilde{\Pi}^{ru}\}(\tau,\vec{\sigma}) + \\ + \int d^3\sigma\{\epsilon k N[\sqrt{\gamma}\, {}^3g^{rs}({}^3\Gamma^u_{rv}\, {}^3\Gamma^v_{su} - {}^3\Gamma^u_{rs}\, {}^3\Gamma^v_{vu}) - \\ - \frac{1}{2k\sqrt{\gamma}}{}^3G_{rsuv}\, {}^3\tilde{\Pi}^{rs}\, {}^3\tilde{\Pi}^{uv}] - \epsilon k\partial_u N\sqrt{\gamma}\, {}^3g^{rs}\, {}^3g^{uv}(\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs}) - \\ -2N_r\, {}^3\Gamma^r_{su}\, {}^3\tilde{\Pi}^{su} + 2\partial_u N_r\, {}^3\tilde{\Pi}^{ru}\}(\tau,\vec{\sigma}). \qquad (A4)$$

In Ref. [6] it is noted that, with the boundary conditions of Refs. [5,6] given in Section III, the term in $\partial_u N$ in the volume integral diverges; the following (non-tensorial) regularization is proposed: $\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs} = \partial_r({}^3g_{vs} - \delta_{vs}) - \partial_v({}^3g_{rs} - \delta_{rs})$ [it may be thought as a kind of subtraction of a static background metric in the spirit of Ref. [42]; in this spirit one could think to use static background metrics ${}^3f_{rs}$ different from $\delta_{rs}$: $\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs} \mapsto \partial_r({}^3g_{vs} - {}^3f_{vs}) - \partial_v({}^3g_{rs} - {}^3f_{rs}) \neq \partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs}$]. If we make a further integration by parts of the volume term containing $\partial_u N$, we get the identity

$$-\int_{S^2_{\tau,\infty}} d^2\Sigma_u\{\epsilon k\sqrt{\gamma}\, {}^3g^{uv}\, {}^3g^{rs}[N(\partial_r\, {}^3g_{vs} - \partial_v\, {}^3g_{rs}) + \partial_u N({}^3g_{rs} - \delta_{rs}) - \\ -\partial_r N({}^3g_{sv} - \delta_{sv})] - 2N_r\, {}^3\tilde{\Pi}^{ru}\}(\tau,\vec{\sigma}) + \\ + \int d^3\sigma[N\tilde{\mathcal{H}} + N_r\, {}^3\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) =$$

$$= + \int d^3\sigma \{ \epsilon N [k \sqrt{\gamma}\, {}^3g^{rs} ({}^3\Gamma^u_{rv}\, {}^3\Gamma^v_{su} - {}^3\Gamma^u_{rs}\, {}^3\Gamma^v_{vu}) - \frac{1}{2k\sqrt{\gamma}} {}^3G_{rsuv}\, {}^3\tilde{\Pi}^{rs}\, {}^3\tilde{\Pi}^{uv}] +$$
$$+ \epsilon k ({}^3g_{vs} - \delta_{vs}) \partial_r [\sqrt{\gamma} \partial_u N ({}^3g^{rs}\, {}^3g^{uv} - {}^3g^{ru}\, {}^3g^{sv})] -$$
$$-2 N_r\, {}^3\Gamma^r_{su}\, {}^3\tilde{\Pi}^{su} + 2 \partial_u N_r\, {}^3\tilde{\Pi}^{ru} \} (\tau, \vec{\sigma}). \qquad \text{(A5)}$$

As shown in Eqs.(11), due to the splitting (10), which explicitly separates the asymptotic part of the lapse and shift functions, the first member of this equation is equal to

$\tilde{\lambda}_A(\tau) P^A_{ADM} + \frac{1}{2} \tilde{\lambda}_{AB}(\tau) J^{AB}_{ADM} + \int d^3\sigma [N \tilde{\mathcal{H}} + N_r\, {}^3\tilde{\mathcal{H}}^r](\tau, \vec{\sigma})$,

with $P^A_{ADM}$ and $J^{AB}_{ADM}$ being the "strong improper Poincaré charges" given by the surface integrals at spatial infinity of Eqs.(12).

This terminology derives from Ref. [40] (see also Appendix D for the treatment of the Hilbert action), where there is the definition of the weak and strong improper conserved non-Abelian charges in Yang-Mills theory and their derivation from the Noether identities implied by the second Noether theorem. In this case one gets (see Ref. [35] c) for the general theory):
i) "strong conserved improper currents" (their conservation is an identity independent from the Euler-Lagrange equations), whose "strong conserved improper charges" are just surface integrals at spatial infinity like in Eqs.(12);
ii) "weak conserved improper currents" (their conservation implies the Euler-Lagrange equations; it is a form of first Noether theorem hidden in the second one), whose "weak conserved improper charges" are volume integrals, like the ones in the second side of Eq.(A5) and in Eqs.(14);
iii) the two kinds of charges differ by the Gauss law first class constraints [like in Eq.(A5)] and, therefore, coincide when use is done of the acceleration-independent Euler-Lagrange equations [i.e. the secondary first class Gauss law constraints like it happens in Eq.(A5)].

In ADM metric gravity it is difficult to check explicitly these statements, because it is expressed in terms of lapse and shift functions and not in terms of their splitted version of Eqs.(10). In this paper we shall adopt the terminology "strong" and "weak" Poincaré charges to refer to surface and volume integrals respectively, even if the strong charges are not strongly conserved improper charges but only weakly conserved ones like the weak charges.

Let us now study the Noether identities [35,40] produced by the second Noether theorem, which can be obtained from the quasi-invariance of the ADM action $S_{ADM}$ under the gauge transformations generated by the first class constraints, as it happens in the Yang-Mills case [40].

From Section V of I we know that the ADM Lagrangian density is $\mathcal{L}_{ADM}(\tau, \vec{\sigma}) = -\epsilon k (N \sqrt{\gamma} [{}^3R + {}^3K^{rs}\, {}^3K_{rs} - ({}^3K)^2])(\tau, \vec{\sigma})$. We shall use the notation

$\delta_o f(\sigma^A) = \bar{f}(\sigma^A) - f(\sigma^A)$, $\partial_B \delta_o f(\sigma^A) = \delta_o \partial_B f(\sigma^A)$,

with $\sigma^A = (\tau; \vec{\sigma})$ [instead one has $\delta f(\sigma^A) = \bar{f}(\bar{\sigma}^A) - f(\sigma^A) = \delta_o f(\sigma^A) + \partial_B f(\sigma^A) \delta\sigma^B$, if $\bar{\sigma}^B = \sigma^B + \delta\sigma^B(\sigma^A)$ with $\delta\sigma^B \neq 0$]. The form of the general variation of the Lagrangian

density is [ $\mathcal{L}_{ADM}$ is considered a function of the following fields: i) $N$ with dependence on $N$, $\partial_\tau N$, $\partial_r N$; ii) $N_s$ with dependence on $N_s$, $\partial_\tau N_s$, $\partial_r N_s$; iii) ${}^3g_{rs}$ with dependence on ${}^3g_{rs}$, $\partial_\tau\, {}^3g_{rs}$, $\partial_u\, {}^3g_{rs}$, $\partial_u\partial_v\, {}^3g_{rs}$]

$$
\begin{aligned}
&\delta_o\mathcal{L}_{ADM}(\tau,\vec{\sigma}) =\\
&= \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial N(\tau,\vec{\sigma})}\delta_o N(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau N(\tau,\vec{\sigma})}\delta_o\partial_\tau N(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N(\tau,\vec{\sigma})}\delta_o\partial_r N(\tau,\vec{\sigma}) +\\
&+\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial N_s(\tau,\vec{\sigma})}\delta_o N_s(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau N_s(\tau,\vec{\sigma})}\delta_o\partial_\tau N_s(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta_o\partial_r N_s(\tau,\vec{\sigma}) +\\
&+\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\partial_\tau\, {}^3g_{uv}(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma}) +\\
&+\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma}) =\\
&= \delta_o N(\tau,\vec{\sigma})\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial N(\tau,\vec{\sigma})} - \partial_\tau\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau N(\tau,\vec{\sigma})} - \partial_r\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N(\tau,\vec{\sigma})}\Big) +\\
&+\delta_o N_s(\tau,\vec{\sigma})\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial N_s(\tau,\vec{\sigma})} - \partial_\tau\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau N_s(\tau,\vec{\sigma})} - \partial_r\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\Big) +\\
&+\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})[\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\, {}^3g_{uv}(\tau,\vec{\sigma})} - \partial_\tau\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau\, {}^3g_{uv}(\tau,\vec{\sigma})} - \partial_r\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} +\\
&+\partial_r\partial_s\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}] + \partial_\tau[\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau\, N(\tau,\vec{\sigma})}\delta_o N(\tau,\vec{\sigma}) +\\
&+\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau\, N_r(\tau,\vec{\sigma})}\delta_o N_r(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_\tau\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})] +\\
&+\partial_r\Big[\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta_o N_s(\tau,\vec{\sigma}) +\\
&+\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r\partial_s\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big) =\\
&= \delta_o N(\tau,\vec{\sigma})L_N(\tau,\vec{\sigma}) + \delta_o N_s(\tau,\vec{\sigma})L^s_{\vec{N}}(\tau,\vec{\sigma}) + \delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})L^{uv}_g(\tau,\vec{\sigma}) +\\
&+\partial_\tau\Big[\tilde{\pi}_N(\tau,\vec{\sigma})\delta_o N(\tau,\vec{\sigma}) + \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})\delta_o N_r + {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma})\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r\Big[\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta_o N_s(\tau,\vec{\sigma}) + \Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r\partial_s\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big),
\end{aligned}
\tag{A6}
$$

where $L_N(\tau,\vec{\sigma}) \,\hat{=}\, 0$, $L^s_{\vec{N}}(\tau,\vec{\sigma}) \,\hat{=}\, 0$ [i.e. $\tilde{\mathcal{H}}(\tau,\vec{\sigma}) \approx 0$, ${}^3\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) \approx 0$], $L^{uv}_g(\tau,\vec{\sigma}) \,\hat{=}\, 0$, are the Euler-Lagrange equations given in Eqs.(71) of I.

By using

$$
\delta_o\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\, {}^3g^{rs}\delta_o\, {}^3g_{rs}, \qquad \delta_o\frac{1}{\sqrt{\gamma}} = -\frac{1}{2\sqrt{\gamma}}{}^3g^{rs}\delta_o\, {}^3g_{rs},
$$

$$
\begin{aligned}
&\delta_o\, {}^3g^{rs} = -{}^3g^{ru}\, {}^3g^{sv}\delta_o\, {}^3g_{uv},\\
&\delta_o\, {}^3\Gamma^l_{uv} = -{}^3g^{lr}\, {}^3\Gamma^s_{uv}\delta_o\, {}^3g_{rs} + \frac{1}{2}[{}^3g^{lr}(\delta^n_u\delta^s_v + \delta^n_v\delta^s_u) - {}^3g^{ln}\delta^r_u\delta^s_v]\partial_n\delta_o\, {}^3g_{rs},\\
&\delta_o(\sqrt{\gamma}\, {}^3R) = -\sqrt{\gamma}({}^3R^{rs} - \frac{1}{2}\, {}^3g^{rs}\, {}^3R)\delta_o\, {}^3g_{rs} +\\
&+\sqrt{\gamma}[{}^3g_{rs}\delta_o\, {}^3g^{rs|u} - \delta_o\, {}^3g^{ru}{}_{|r}]_{|u},\\
&\delta_o\, {}^3K_{rs} = -\frac{1}{N}\, {}^3K_{rs}\delta_oN + \frac{1}{2N}\{(\delta_oN_u)_{|v} + (\delta_oN_v)_{|u} + 2N^r\, {}^3\Gamma^s_{uv}\delta_o\, {}^3g_{rs} -\\
&-[N^r(\delta^n_u\delta^s_v + \delta^n_v\delta^s_u) - N^n\delta^r_u\delta^s_v]\partial_n\delta_o\, {}^3g_{rs} - \partial_\tau\delta_o\, {}^3g_{rs}\},\\
&\delta_o\, {}^3K = -\frac{1}{N}{}^3K\delta_oN + \frac{1}{N}{}^3g^{rs}(\delta_oN_r)_{|s} - [{}^3K^{rs} - \frac{N^r}{N}{}^3g^{uv}\, {}^3\Gamma^s_{uv}]\delta_o\, {}^3g_{rs} -\\
&-\frac{1}{2N}(2N^r\, {}^3g^{ns} - N^n\, {}^3g^{rs})\partial_n\delta_o\, {}^3g_{rs} - \frac{1}{2N}{}^3g^{rs}\partial_\tau\delta_o\, {}^3g_{rs}, \qquad \text{(A7)}
\end{aligned}
$$

we get the following form for explicit the variation of the ADM Lagrangian density

$$
\begin{aligned}
\delta_o\mathcal{L}_{ADM}(\tau,\vec{\sigma}) &= -\epsilon k\{\sqrt{\gamma}[{}^3R + {}^3K_{rs}\, {}^3K^{rs} - ({}^3K)^2]\delta_oN + N\delta_o(\sqrt{\gamma}\, {}^3R) +\\
&+ N[{}^3K_{rs}\, {}^3K^{rs} - ({}^3K)^2]\delta_o\sqrt{\gamma} + 2N\sqrt{\gamma}[{}^3K^{rs}\delta_o\, {}^3K_{rs} - {}^3K\delta_o\, {}^3K]\} =\\
&= -\tilde{\mathcal{H}}\delta_oN - 2\, {}^3\tilde{\Pi}^{rs}(\delta_oN_r)_{|s} - N\{\epsilon k\sqrt{\gamma}({}^3R^{rs} - \frac{1}{2}{}^3g^{rs}\, {}^3R) +\\
&+ 2(N^r\, {}^3\Gamma^s_{uv}\, {}^3\tilde{\Pi}^{uv} - {}^3K^r{}_m\, {}^3\tilde{\Pi}^{ms}) + \frac{1}{2}\epsilon k\sqrt{\gamma}({}^3K^{uv}\, {}^3K_{uv} - ({}^3K)^2){}^3g^{rs}\}\delta_o\, {}^3g_{rs} +\\
&+ (N^n\, {}^3\tilde{\Pi}^{rs} - 2N^r\, {}^3\tilde{\Pi}^{ns})\partial_n\delta_o\, {}^3g_{rs} + {}^3\tilde{\Pi}^{rs}\partial_\tau\delta_o\, {}^3g_{rs} -\\
&\quad -\epsilon kN\partial_l\{\sqrt{\gamma}[\frac{1}{2}\Big({}^3g^{rs}\, {}^3g^{uv}\, {}^3g^{lm} - 2({}^3g^{ur}\, {}^3g^{vs}\, {}^3g^{lm} + {}^3g^{uv}\, {}^3g^{lr}\, {}^3g^{ms})\Big)\cdot\\
&\quad (\partial_v\, {}^3g_{um} - \partial_m\, {}^3g_{uv})\delta_o\, {}^3g_{rs} + ({}^3g^{rn}\, {}^3g^{ls} - {}^3g^{rs}\, {}^3g^{ln})\partial_n\delta_o\, {}^3g_{rs}]\}, \qquad \text{(A8)}
\end{aligned}
$$

We want to study the invariance properties of $S_{ADM} = \int d\tau d^3\sigma\mathcal{L}_{ADM}(\tau,\vec{\sigma})$ under the most general gauge transformations generated by the first class constraints, namely generated by

$$
G[\alpha,\alpha_r,\lambda_N,\lambda^{\vec{N}}_r] = \int d^3\sigma[\alpha\tilde{\mathcal{H}} + \alpha_r\, {}^3\tilde{\mathcal{H}}^r + \lambda_N\tilde{\pi}^N + \lambda^{\vec{N}}_r\tilde{\pi}^r_{\vec{N}}](\tau,\vec{\sigma}).
$$

They produce the following variations of the Lagrangian variables

$$
\begin{aligned}
&\delta_oN(\tau,\vec{\sigma}) = \{N(\tau,\vec{\sigma}),G\} = \lambda_N(\tau,\vec{\sigma}) \,{\buildrel \circ \over =}\, \partial_\tau N(\tau,\vec{\sigma}),\\
&\delta_oN_r(\tau,\vec{\sigma}) = \{N_r(\tau,\vec{\sigma}),G\} = \lambda^{\vec{N}}_r(\tau,\vec{\sigma}) \,{\buildrel \circ \over =}\, \partial_\tau N_r(\tau,\vec{\sigma}),\\
&\delta_o\, {}^3g_{rs}(\tau,\vec{\sigma}) = \{{}^3g(\tau,\vec{\sigma}),G\} = [\alpha_{r|s} + \alpha_{s|r} - 2\alpha\, {}^3K_{rs}](\tau,\vec{\sigma}) =\\
&= \Big(\alpha_{r|s} + \alpha_{s|r} - \frac{\alpha}{N}[N_{r|s} + N_{s|r} - \partial_\tau\, {}^3g_{rs}]\Big)(\tau,\vec{\sigma}), \qquad \text{(A9)}
\end{aligned}
$$

One sees that on the solutions of the Hamilton-Dirac equations one has $\lambda_N \,{\buildrel \circ \over =}\, \partial_\tau N = \partial_\tau\alpha$, $\lambda^{\vec{N}}_r \,{\buildrel \circ \over =}\, \partial_\tau N_r = \partial_\tau\alpha_r$, if $\alpha = N$, $\alpha_r = N_r$, as it happens if G is identified with the Dirac Hamiltonian.

Since it is difficult to find the associated quasi-invariances directly, let us look at the various Noether transformations separately.

1) Under the variations generated by

$G_1[\lambda_N, \lambda^{\vec N}_r] = \int d^3\sigma [\lambda_N \tilde \pi^N + \lambda^{\vec N}_r \tilde \pi^r_{\vec N}](\tau ,\vec \sigma ) = G[0,0,\lambda_N, \lambda^{\vec N}_r]$

one gets the Noether identities

$$
\begin{aligned}
\delta_o {\cal L}_{ADM}(\tau ,\vec \sigma ) &= \lambda_N(\tau ,\vec \sigma ) L_N(\tau ,\vec \sigma ) + \lambda^{\vec N}_s(\tau ,\vec \sigma ) L^s_{\vec N}(\tau ,\vec \sigma ) +\\
&+ \partial_{\tau} [\tilde \pi_N(\tau ,\vec \sigma ) \lambda_N(\tau ,\vec \sigma ) + \tilde \pi^r_{\vec N}(\tau ,\vec \sigma ) \lambda^{\vec N}_r(\tau ,\vec \sigma )] +\\
&+ \partial_r [{{\partial {\cal L}_{ADM}(\tau ,\vec \sigma )}\over {\partial \partial_r N_s(\tau ,\vec \sigma )}} \lambda^{\vec N}_s(\tau ,\vec \sigma )] \equiv\\
&\equiv - \Big( \lambda_N \tilde {\cal H} + \lambda^{\vec N}_r \tilde {\cal H}^r + \partial_r [2\, {}^3\tilde \Pi^{rs} \lambda^{\vec N}_s] \Big) (\tau ,\vec \sigma ) \,{\buildrel \circ \over =}\\
&{\buildrel \circ \over =}\, - \partial_r [2\, {}^3\tilde \Pi^{rs} \lambda^{\vec N}_s](\tau ,\vec \sigma ),
\end{aligned}
\qquad \text{(A10)}
$$

by using the definition of the ADM momentum and the acceleration independent Euler-Lagrange equations corresponding to $\tilde {\cal H}(\tau ,\vec \sigma ) \approx 0$, $\tilde {\cal H}^r(\tau ,\vec \sigma ) \approx 0$ [see Eqs.(70) of I].

By equating the coefficients of $\partial_{\tau} \lambda_N$, $\lambda_N$ and of $\partial_{\tau} \lambda^{\vec N}_r$, $\partial_s \lambda^{\vec N}_r$, $\lambda^{\vec N}_r$ on the two sides of the previous identity one gets the following Noether identities equivalent to the primary and secondary constraints in the Hamiltonian formulation

$$
\begin{aligned}
&\tilde \pi_N \equiv 0,\\
&0 \equiv \partial_{\tau} \tilde \pi_N \equiv - \tilde {\cal H} - L_N \qquad \Rightarrow \tilde {\cal H} \equiv - L_N \,{\buildrel \circ \over =}\, 0,
\end{aligned}
\qquad \text{(A11)}
$$

$$
\begin{aligned}
&{{\partial {\cal L}_{ADM}}\over {\partial \partial_r N_s}} \equiv -2\, {}^3\tilde \Pi^{rs} = -2 {{\partial {\cal L}_{ADM}}\over {\partial \partial_{\tau}\, {}^3g_{rs}}},\\
&\tilde \pi^r_{\vec N} \equiv 0,\\
&0 \equiv \partial_{\tau} \tilde \pi^r_{\vec N} \equiv - \tilde {\cal H}^r - L^r_{\vec N} - \partial_s (2\, {}^3\tilde \Pi^{rs} + {{\partial {\cal L}_{ADM}}\over {\partial \partial_r N_s}}) \equiv\\
&\equiv - \tilde {\cal H}^r - L^r_{\vec N} \qquad \Rightarrow \tilde {\cal H}^r \equiv - L^r_{\vec N} \,{\buildrel \circ \over =}\, 0.
\end{aligned}
\qquad \text{(A12)}
$$

2) Since $S_{ADM} = \int d\tau d^3\sigma {\cal L}_{ADM}(\tau ,\vec \sigma )$ is invariant under 3-diffeomorphisms of $Diff\, \Sigma_{\tau}$ [$d^3\sigma {\cal L}_{ADM}(\tau ,\vec \sigma )$ is a scalar under $\sigma^r \mapsto \bar \sigma^r = \sigma^r + \delta \sigma^r(\vec \sigma )$; see Eqs.(31) of I for the following formulas], under the variations

$$
\begin{aligned}
\delta \sigma^r &= \xi^r(\vec \sigma ),\\
\delta^D_o N(\tau ,\vec \sigma ) &= - \xi^r(\vec \sigma ) \partial_r N(\tau ,\vec \sigma ),\\
\delta^D_o N^r(\tau ,\vec \sigma ) &= [\partial_s \xi^r(\vec \sigma ) - \delta^r_s \xi^u(\vec \sigma ) \partial_u ] N^s(\tau ,\vec \sigma ) = {\cal L}_{-\xi^t\partial_t} N^r(\tau ,\vec \sigma ),\\
\delta^D_o N_r(\tau ,\vec \sigma ) &= - [\partial_r \xi^s(\vec \sigma ) + \delta^s_r \xi^u(\vec \sigma ) \partial_u ] N_s(\tau ,\vec \sigma ),\\
\delta^D_o\, {}^3g_{rs}(\tau ,\vec \sigma ) &= - [\delta^u_r \partial_s \xi^v(\vec \sigma ) + \delta^v_s \partial_r \xi^u(\vec \sigma ) + \delta^u_r \delta^v_s \xi^t(\vec \sigma ) \partial_t ] {}^3g_{uv}(\tau ,\vec \sigma ) =\\
&= - [\xi_{r|s} + \xi_{s|r}](\tau ,\vec \sigma ) = {\cal L}_{-\xi^t\partial_t}\, {}^3g_{rs}(\tau ,\vec \sigma ) =\\
&= \{ {}^3g_{rs}(\tau ,\vec \sigma ), -\int d^3\sigma^{'} \xi_u(\vec \sigma ) \tilde {\cal H}^u(\tau ,{\vec \sigma}^{'}) \},\\
&\Downarrow\\
\delta^D N(\tau ,\vec \sigma ) &= \delta^D_o N(\tau ,\vec \sigma ) + \xi^r(\vec \sigma ) \partial_r N(\tau ,\vec \sigma ) = 0,\\
\delta^D N^r(\tau ,\vec \sigma ) &= \delta^D_o N^r(\tau ,\vec \sigma ) + \xi^u(\vec \sigma ) \partial_u N^r(\tau ,\vec \sigma ),\\
\delta^{D3}g_{rs}(\tau ,\vec \sigma ) &= \delta^{D3}_o g_{rs}(\tau ,\vec \sigma ) + \xi^u(\vec \sigma ) \partial_u\, {}^3g_{rs}(\tau ,\vec \sigma ),
\end{aligned}
\qquad \text{(A13)}
$$

we get the following Noether identity

$$
\begin{aligned}
&\delta^D_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) =\\
&= \delta^D_o N(\tau,\vec{\sigma}) L_N(\tau,\vec{\sigma}) + \delta^D_o N_s(\tau,\vec{\sigma}) L^s_{\vec{N}}(\tau,\vec{\sigma}) + \delta^D_o\, {}^3g_{uv}(\tau,\vec{\sigma}) L^{uv}_g(\tau,\vec{\sigma}) +\\
&+\partial_\tau \Big[\tilde{\pi}^N(\tau,\vec{\sigma})\delta^D_o N(\tau,\vec{\sigma}) + \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})\delta^D_o N_r(\tau,\vec{\sigma}) + {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma})\delta^D_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r \Big[\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta^D_o N_s(\tau,\vec{\sigma}) + \Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s \frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)\delta^D_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r\partial_s \Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta^D_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big) +\\
&+\partial_r[\mathcal{L}_{ADM}\xi^r](\tau,\vec{\sigma}) \equiv 0.
\end{aligned}
\tag{A14}
$$

As shown in the paper c) in Ref. [35], the same Noether identities can be obtained with the transformations

$$\delta_o \sigma^r = 0,\ \delta_o N = \delta^D_o N,\ \delta_o N_r = \delta^D_o N_r,\ \delta_o\, {}^3g_{rs} = \delta^D_o\, {}^3g_{rs},$$

since now we get

$$
\begin{aligned}
&\delta_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) =\\
&= \delta_o N(\tau,\vec{\sigma}) L_N(\tau,\vec{\sigma}) + \delta_o N_s(\tau,\vec{\sigma}) L^s_{\vec{N}}(\tau,\vec{\sigma}) + \delta_o\, {}^3g_{uv}(\tau,\vec{\sigma}) L^{uv}_g(\tau,\vec{\sigma}) +\\
&+\partial_\tau \Big[\tilde{\pi}^N(\tau,\vec{\sigma})\delta_o N(\tau,\vec{\sigma}) + \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})\delta_o N_r(\tau,\vec{\sigma}) + {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma})\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r \Big[\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta_o N_s(\tau,\vec{\sigma}) + \Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s \frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] +\\
&+\partial_r\partial_s \Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big) \equiv\\
&\equiv -\partial_r(\mathcal{L}_{ADM}\xi^r)(\tau,\vec{\sigma}).
\end{aligned}
\tag{A15}
$$

If the 3-diffeomorphisms are $\tau$-dependent, namely in Eqs.(A13) the variations $\delta^D_o$ are replaced by $\delta^{D\tau}_o$ with $\delta\sigma^r = \xi^r(\tau,\vec{\sigma})$, one has

$$
\begin{aligned}
\delta^{D\tau}_o \partial_\tau\, {}^3g_{rs} &= \partial_\tau \delta^{D\tau}_o\, {}^3g_{rs} = \delta^D_o \partial_\tau\, {}^3g_{rs} + \delta^{'} \partial_\tau\, {}^3g_{rs} =\\
&= \delta^D_o \partial_\tau\, {}^3g_{rs} - [\delta^u_r \partial_s\partial_\tau \xi^v + \delta^v_s \partial_r\partial_\tau\xi^u + \delta^u_r\delta^v_s \partial_\tau\xi^t\partial_t]^3g_{uv},\\
&\Downarrow\\
\delta^{D\tau}_o\, {}^3K_{rs} &= \delta^D_o\, {}^3K_{rs} + \delta^{'}\, {}^3K_{rs} =\\
&= \delta^D_o\, {}^3K_{rs} + \frac{1}{2N}[\delta^u_r \partial_s\partial_\tau \xi^v + \delta^v_s \partial_r\partial_\tau\xi^u + \delta^u_r\delta^v_s \partial_\tau\xi^t\partial_t]^3g_{uv},\\
&\Downarrow\\
\delta^{D\tau}_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) &= \delta_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) - [2\epsilon k N\sqrt{\gamma}({}^3K^{rs} - {}^3K\, {}^3g^{rs})\delta^{'}\, {}^3K_{rs}](\tau,\vec{\sigma}) \equiv\\
&\equiv -\partial_r(\mathcal{L}_{ADM}\xi^r)(\tau,\vec{\sigma}) - [{}^3\tilde{\Pi}^{rs}\delta^{'}\, {}^3K_{rs}](\tau,\vec{\sigma}) =\\
&= -\partial_r(\mathcal{L}_{ADM}\xi^r)(\tau,\vec{\sigma}) - 2[{}^3g_{rv}\, {}^3\tilde{\Pi}^{rs}\partial_s\partial_\tau\xi^v](\tau,\vec{\sigma}) -\\
&- [{}^3\tilde{\Pi}^{rs}\partial_u\, {}^3g_{rs}\partial_\tau\xi^u](\tau,\vec{\sigma}) =\\
&= \delta^D_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) - [\partial_\tau\xi^u\, {}^3g_{ur}\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) -\\
&- \partial_s[2\,{}^3g_{ru}\, {}^3\tilde{\Pi}^{rs}\partial_\tau\xi^u](\tau,\vec{\sigma}).
\end{aligned}
\tag{A16}
$$

Therefore, for

$\delta\sigma^r = 0,\ \delta_o N = \delta_o^{D\tau} N,\ \delta_o N_r = \delta_o^{D\tau} N_r,\ \delta_o\, {}^3g_{rs} = \delta_o^{D\tau}\, {}^3g_{rs},$

we get

$$
\begin{aligned}
&\delta_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) = \\
&= \delta_o N(\tau,\vec{\sigma}) L_N(\tau,\vec{\sigma}) + \delta_o N_s(\tau,\vec{\sigma}) L^s_{\vec{N}}(\tau,\vec{\sigma}) + \delta_o\, {}^3g_{uv}(\tau,\vec{\sigma}) L_g^{uv}(\tau,\vec{\sigma}) + \\
&+\partial_\tau \Big[\tilde{\pi}^N(\tau,\vec{\sigma})\delta_o N(\tau,\vec{\sigma}) + \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})\delta_o N_r(\tau,\vec{\sigma}) + {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma})\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] + \\
&+\partial_r\Big[\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta_o N_s(\tau,\vec{\sigma}) + \Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s \frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] + \\
&+\partial_r\partial_s\Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta_o\, {}^3g_{uv}(\tau,\vec{\sigma})\Big) \equiv \\
&\equiv -\partial_s[2\, {}^3g_{ru}\, {}^3\tilde{\Pi}^{rs}\partial_\tau\xi^u + \mathcal{L}_{ADM}\xi^s](\tau,\vec{\sigma}) - [\partial_\tau\xi^u\, {}^3g_{ur}\tilde{\mathcal{H}}^r](\tau,\vec{\sigma}) \,\circeq \\
&\circeq\, - \partial_s[2\, {}^3g_{ru}\, {}^3\tilde{\Pi}^{rs}\partial_\tau\xi^u + \mathcal{L}_{ADM}\xi^s](\tau,\vec{\sigma})],
\end{aligned} \tag{A17}
$$

by using the acceleration independent Euler-Lagrange equations corresponding to $\tilde{\mathcal{H}}^r(\tau,\vec{\sigma}) \approx 0$.

Therefore, if we choose $\alpha_r(\tau,\vec{\sigma}) = -\xi_r(\tau,\vec{\sigma})$ $[\alpha^r = {}^3g^{rs}\alpha_s]$, the generator

$$
\begin{aligned}
G_2[\alpha_r] &= \int d^3\sigma[\alpha_r\tilde{\mathcal{H}}^r + \alpha^r\partial_r N\tilde{\pi}^N + [\partial_r\alpha^s + \delta^s_r\alpha^u\partial_u]N_s\tilde{\pi}^r_{\vec{N}}](\tau,\vec{\sigma}) = \\
&= G[0, \alpha_r, \alpha^r\partial_r N, (\partial_r\alpha^s + \delta^s_r\alpha^u\partial_u)N_s],
\end{aligned} \tag{A18}
$$

yields the previous transformations.

By putting together Eqs.(A10) and (A16), we get that the gauge transformations with generator

$$
\begin{aligned}
G_3[\alpha_r, \lambda_N, \lambda_r^{\vec{N}}] &= \int d^3\sigma[\alpha_r\tilde{\mathcal{H}}^r + \lambda_N\tilde{\pi}^N + \lambda_r^{\vec{N}}\tilde{\pi}^r_{\vec{N}}](\tau,\vec{\sigma}) = \\
&= G[0, \alpha_r, \lambda_N, \lambda_r^{\vec{N}}] = G_2[\alpha_r] + \\
&+ G_1[\lambda_N - \alpha^r\partial_r N, \lambda_r^{\vec{N}} - (\partial_r\alpha^s + \delta^s_r\alpha^u\partial_u)N_s], \\
&\quad \Downarrow \\
&\quad \delta_o N = \lambda_N, \qquad \delta_o N_r = \lambda_r^{\vec{N}}, \qquad \delta_o\, {}^3g_{rs} = \alpha_{r|s} + \alpha_{s|r},
\end{aligned} \tag{A19}
$$

yields the Noether identities

$$
\begin{aligned}
&\delta_o \mathcal{L}_{ADM}(\tau,\vec{\sigma}) = \\
&= \lambda_N(\tau,\vec{\sigma}) L_N(\tau,\vec{\sigma}) + \lambda_s^{\vec{N}}(\tau,\vec{\sigma}) L^s_{\vec{N}}(\tau,\vec{\sigma}) + (\alpha_{u|v} + \alpha_{v|u})(\tau,\vec{\sigma}) L_g^{uv}(\tau,\vec{\sigma}) + \\
&+\partial_\tau\Big[\tilde{\pi}^N(\tau,\vec{\sigma})\lambda_N(\tau,\vec{\sigma}) + \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})\lambda_r^{\vec{N}}(\tau,\vec{\sigma}) + {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma})(\alpha_{u|v} + \alpha_{v|u})(\tau,\vec{\sigma})\Big] + \\
&+\partial_r\Big[\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\lambda_s^{\vec{N}}(\tau,\vec{\sigma}) + \\
&+\Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)(\alpha_{u|v} + \alpha_{v|u})(\tau,\vec{\sigma})\Big] +
\end{aligned}
$$

$$
\begin{aligned}
&+\partial_r\partial_s\Big(\frac{\partial \mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}(\alpha_{u|v}+\alpha_{v|u})(\tau,\vec{\sigma})\Big) \equiv\\
&\equiv -\Big([\lambda_N - \alpha^r\partial_r N]\tilde{\mathcal{H}}\Big)(\tau,\vec{\sigma}) -\\
&-\Big([\lambda^{\vec{N}}_r - (\partial_r\alpha^s + \delta^s_r\alpha^u\partial_u)N_s - {}^3g_{ru}\partial_\tau\alpha^u]\tilde{\mathcal{H}}^r\Big)(\tau,\vec{\sigma}) -\\
&-\partial_r\Big(2\,{}^3\tilde{\Pi}^{rs}[\lambda^{\vec{N}}_s - (\partial_s\alpha^v + \delta^v_s\alpha^u\partial_u)N_v] -\\
&-[2\,{}^3\tilde{\Pi}^{rs}\,{}^3g_{su}\partial_\tau\alpha^u + \mathcal{L}_{ADM}\alpha^r]\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{A20}
$$

By putting $\lambda_N = 0$ and $\lambda^{\vec{N}}_r = \partial_\tau\alpha_r \,\circeq\, \partial_\tau N_r$ (so that the generator is $G[0,\alpha_r,0,\partial_\tau\alpha_r]$ and one has $\alpha_r \,\circeq\, N_r$), one recovers the Noether identities of Eqs.(A12) and, in addition, new identities. Indeed, by equating the coefficients of $\partial_r\partial_\tau\alpha_s$, $\partial^2_\tau\alpha_r$, $\partial_\tau\alpha_r$, $\partial_r\partial_s\partial_u\alpha_v$, $\partial_r\partial_s\alpha_t$, $\partial_r\alpha_s$, $\alpha_t$ on both sides we get 7 Noether identities [the first three reproduce Eqs.(A12)]

$$
\begin{aligned}
&\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r N_s} + 2\,{}^3\tilde{\Pi}^{rs} \equiv 0,\\
&\tilde{\pi}^r_{\vec{N}} \equiv 0,\\
&\partial_\tau\tilde{\pi}^r_{\vec{N}} + \tilde{\mathcal{H}}^r + L^r_{\vec{N}} \equiv 0, \quad \Rightarrow \quad \tilde{\mathcal{H}}^r \equiv -L^r_{\vec{N}},\\
&\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_{(r}\partial_s\,{}^3g_{u)v}} \equiv 0,\\
&\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_{(r}\,{}^3g_{s)t}} - {}^3\Gamma^t_{uv}\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\partial_s\,{}^3g_{uv}} - {}^3\tilde{\Pi}^{rs}\,{}^3g^{tu}N_u \equiv 0,\\
&2L^{rs}_g - N_v\,{}^3g^{v(r}\tilde{\mathcal{H}}^{s)} + 2[\partial_\tau{}^3\tilde{\Pi}^{rs} - {}^3g_{tu}\,{}^3\tilde{\Pi}^{t(r}\partial_\tau\,{}^3g^{s)u}] -\\
&-2\,{}^3g^{v(s}\partial_t({}^3\tilde{\Pi}^{r)t}N_v) - 4N_v\,{}^3\tilde{\pi}^{t(r}\partial_t\,{}^3g^{s)v} - 2\,{}^3\tilde{\Pi}^{v(r}\,{}^3g^{s)u}\partial_u N_v - {}^3g^{rs}\mathcal{L}_{ADM} +\\
&+2\partial_t(\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_t\,{}^3g_{rs}} - \partial_u\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_u\partial_t\,{}^3g_{rs}}) - 2\,{}^3\Gamma^{(r}_{uv}\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_{s)}\,{}^3g_{uv}} + 2(\partial_t\,{}^3\Gamma^{(r}_{uv})\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_{s)}\partial_t\,{}^3g_{uv}} \equiv 0,\\
&-2{}^3\Gamma^t_{uv}L^{uv}_g - {}^3g^{tr}\partial_r\tilde{\mathcal{H}} - N_s\partial_r\,{}^3g^{st}\tilde{\mathcal{H}}^r - {}^3g^{tu}\partial_u N_s\tilde{\mathcal{H}}^s +\\
&+\partial_\tau\,{}^3g^{ut}\,{}^3g_{ur}\tilde{\mathcal{H}}^r - 2\partial_\tau({}^3\Gamma^t_{uv}\,{}^3\tilde{\Pi}^{uv}) - 2\partial_r({}^3\tilde{\Pi}^{rs}\,{}^3g_{su})\partial_\tau\,{}^3g^{ut} - 2\,{}^3\tilde{\Pi}^{rs}\,{}^3g_{su}\partial_r\partial_\tau\,{}^3g^{ut} -\\
&-2\partial_r[{}^3\tilde{\Pi}^{rs}(N_v\partial_s\,{}^3g^{vt} + {}^3g^{vt}\partial_v N_s)] - \partial_r({}^3g^{rt}\mathcal{L}_{ADM}) -\\
&-2\partial_r({}^3\Gamma^t_{uv}[\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\,{}^3g_{uv}} - 2\partial_s\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\partial_s\,{}^3g_{uv}}]) - 2\partial_r\partial_s({}^3\Gamma^t_{uv}\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\partial_s\,{}^3g_{uv}}) \equiv 0.
\end{aligned}
\tag{A21}
$$

One can verify that the last two identities are actually $0 \equiv 0$. Suitable combinations of these identities should rebuild three of the contracted Bianchi identities quoted after Eq.(10) of I and allow to rewrite the part of Eq.(A5) connected with the supermomentum constraints as a weak charge (a Noether constant expressed as a volume integral of a charge density) equal (modulo the constraints) to a strong charge (a surface integral of a charge density expressible in terms of a superpotential like in Appendix D). This will be studied elsewhere.

3) To preserve the 3+1 splitting of $M^4$, the ADM action (like the actions of parametrized theories on spacelike hypersurfaces in Minkowski spacetime) is invariant under $\vec{\sigma}$-independent $\tau$-reparametrizations $\tau \mapsto \tau^{'}(\tau) = \tau + \delta\tau(\tau)$ [instead the diffeomorphisms in $Diff\, M^4$ mix all the coordinates $x^\mu$]. Since the lapse and shift functions transform like $\frac{d}{d\tau}$, one has $\delta_\tau S_{ADM} \equiv 0$ [namely $d\tau\mathcal{L}_{ADM}(\tau,\vec{\sigma}) = d\tau N(\tau,\vec{\sigma})\mathcal{L}^{'}_{ADM}(\tau,\vec{\sigma})$ is a scalar] under the "non Lagrangian" $\tau$-reparametrizations

$$
\begin{aligned}
\delta\tau(\tau) &= \bar{\tau}(\tau) - \tau,\\
\delta_\tau N(\tau,\vec{\sigma}) &= -\frac{d\delta\tau}{d\tau} N(\tau,\vec{\sigma}), \qquad \Rightarrow\ \delta_\tau[d\tau N(\tau,\vec{\sigma})] = 0,\\
\delta_\tau N_r(\tau,\vec{\sigma}) &= -\frac{d\delta\tau}{d\tau} N_r(\tau,\vec{\sigma}),\\
\delta_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) &= 0, \qquad \delta_\tau\partial_u\, {}^3g_{rs}(\tau,\vec{\sigma}) = 0, ...\\
\delta_\tau\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) &= -\frac{d\delta\tau}{d\tau}\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}),\\
&\Rightarrow \qquad \delta_\tau\, {}^3K_{rs}(\tau,\vec{\sigma}) = 0,\\
&\qquad\qquad \Downarrow\\
\delta_\tau[d\tau \mathcal{L}_{ADM}(\tau,\vec{\sigma})] &= \delta_\tau[d\tau N(\tau,\vec{\sigma})\mathcal{L}^{'}_{ADM}(\tau,\vec{\sigma})] =\\
&= d\tau N(\tau,\vec{\sigma})\delta_\tau \mathcal{L}^{'}_{ADM}(\tau,\vec{\sigma}) \equiv 0.
\end{aligned} \tag{A22}
$$

Following the paper c) in Ref. [35], one reconstructs a real Noether Lagrangian transformation

$$
\begin{aligned}
&\delta\tau(\tau)\\
\delta_{o\tau} N(\tau,\vec{\sigma}) &= \delta_\tau N(\tau,\vec{\sigma}) = -\frac{d\delta\tau(\tau)}{d\tau} N(\tau,\vec{\sigma}),\\
\delta_{o\tau} N_r(\tau,\vec{\sigma}) &= \delta_\tau N_r(\tau,\vec{\sigma}) = -\frac{d\delta\tau(\tau)}{d\tau} N_r(\tau,\vec{\sigma}),\\
\delta_{o\tau}\, {}^3g_{rs}(\tau,\vec{\sigma}) &= -\delta\tau(\tau)\, \partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}),\\
\delta_{o\tau}\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) &= -\frac{d\delta\tau(\tau)}{d\tau}\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) - \delta\tau(\tau)\, \partial^2_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) =\\
&= \delta_\tau\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}) + \delta^{'}\partial_\tau\, {}^3g_{rs}(\tau,\vec{\sigma}),
\end{aligned} \tag{A23}
$$

Since, with $\mathcal{L}^{'}_{ADM} = \mathcal{L}_{ADM}/N$, we get $\frac{\partial}{\partial N}\mathcal{L}^{'}_{ADM} = \frac{1}{N}(\frac{\partial \mathcal{L}_{ADM}}{\partial N} - \mathcal{L}^{'}_{ADM})$, then by using $\delta_\tau \mathcal{L}^{'}_{ADM} = \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial N}\delta_\tau N + \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial N_r}\delta_\tau N_r + \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial_s N_r}\partial_s\delta_\tau N_r + \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial\partial_\tau\, {}^3g_{rs}}\delta_\tau\partial_\tau\, {}^3g_{rs} \equiv 0$, we obtain

$$
\begin{aligned}
\delta_{o\tau}\mathcal{L}^{'}_{ADM} &= \frac{1}{N}\Big[\delta_{o\tau}\mathcal{L}_{ADM} - \frac{\mathcal{L}_{ADM}}{N}\delta_{o\tau}N\Big] =\\
&= \frac{1}{N}(L_N - \mathcal{L}^{'}_{ADM})\delta_{o\tau}N + \frac{1}{N}L^r_{\vec{N}}\delta_{o\tau}N_r + \frac{1}{N}L^{rs}_g\delta_{o\tau}\, {}^3g_{rs} +\\
&+ \frac{1}{N}\partial_\tau[\tilde{\pi}^N\delta_{o\tau}N + \tilde{\pi}^r_{\vec{N}}\delta_{o\tau}N_r + {}^3\tilde{\Pi}^{rs}\delta_{o\tau}\, {}^3g_{rs}] +\\
&+ \frac{1}{N}\partial_r[\frac{\partial \mathcal{L}_{ADM}}{\partial\partial_r N_s}\delta_{o\tau}N_s + (\frac{\partial \mathcal{L}_{ADM}}{\partial\partial_r\, {}^3g_{uv}} - 2\partial_s\frac{\partial \mathcal{L}_{ADM}}{\partial\partial_r\partial_s\, {}^3g_{uv}})\delta_{o\tau}\, {}^3g_{uv}] +\\
&+ \frac{1}{N}\partial_r\partial_s(\frac{\partial \mathcal{L}_{ADM}}{\partial\partial_r\partial_s\, {}^3g_{uv}}\delta_{o\tau}\, {}^3g_{uv}) \equiv\\
&\equiv \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial\, {}^3g_{uv}}\delta_{o\tau}\, {}^3g_{uv} + \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial\partial_\tau\, {}^3g_{uv}}\delta^{'}\partial_\tau\, {}^3g_{uv} +\\
&+ \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial\partial_r\, {}^3g_{uv}}\delta_{o\tau}\partial_r\, {}^3g_{uv} + \frac{\partial \mathcal{L}^{'}_{ADM}}{\partial\partial_r\partial_s\, {}^3g_{uv}}\delta_{o\tau}\partial_r\partial_s\, {}^3g_{uv},\\
&\Downarrow \qquad by\, putting\ \beta(\tau) = -\delta\tau(\tau)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\ddot{\beta}(\tau)}{N} \quad [\tilde{\pi}^N N + \tilde{\pi}^r_{\vec{N}} N_r] + \\
+&\frac{\dot{\beta}(\tau)}{N} \quad [\partial_\tau(\tilde{\pi}^N N + \tilde{\pi}^r_{\vec{N}} N_r) + L_N N + L^r_{\vec{N}} N_r - \mathcal{L}_{ADM} + \\
&\quad + {}^3\tilde{\Pi}^{rs}\partial_\tau\, {}^3g_{rs} + \partial_r(\frac{\partial \mathcal{L}_{ADM}}{\partial\partial_r N_s} N_s)] + \\
+&\frac{\beta(\tau)}{N} \quad [L^{rs}_g \partial_\tau\, {}^3g_{rs} + \partial_\tau({}^3\tilde{\Pi}^{rs}\partial_\tau\, {}^3g_{rs}) - {}^3\tilde{\Pi}^{rs}\partial^2_\tau\, {}^3g_{rs} + \\
&\quad + \partial_r\Big[(\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\, {}^3g_{uv}} - 2\partial_s \frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\partial_s\, {}^3g_{uv}})\partial_\tau\, {}^3g_{uv}\Big] + \partial_r\partial_s(\frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\partial_s\, {}^3g_{uv}}\partial_\tau\, {}^3g_{uv}) - \\
&\quad - \frac{\partial\mathcal{L}_{ADM}}{\partial\, {}^3g_{uv}}\partial_\tau\, {}^3g_{rs} - \frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\, {}^3g_{uv}}\partial_r\partial_\tau\, {}^3g_{rs} - \frac{\partial\mathcal{L}_{ADM}}{\partial\partial_r\partial_s\, {}^3g_{uv}}\partial_r\partial_s\partial_\tau\, {}^3g_{rs} \equiv 0. \qquad \text{(A24)}
\end{aligned}
$$

As one can check the three identities given by the vanishing of the coefficients of $\ddot{\beta}(\tau)$, $\dot{\beta}(\tau)$, $\beta(\tau)$ [due to Eqs.(77), (80) of I, the second identity is $N\tilde{\mathcal{H}} + N_r\, {}^3\tilde{\mathcal{H}}^r \equiv -NL_N - N_r L^r_{\vec{N}}$] are satisfied and contain implicitly the fourth contracted Bianchi identity and the reformulation of the part of Eq.(A5) connected with the superhamiltonian constraint as a weak charge weakly equal to a strong charge. Again this will be studied elsewhere.

Under the equivalent (see the paper c) in Ref. [35]) Noether transformation

$\delta^{'}_{o\tau} N = \dot{\beta} N$, $\delta^{'}_{o\tau} N_r = \dot{\beta} N_r$, $\delta^{'}_{o\tau}\, {}^3g_{rs} = \beta\partial_\tau\, {}^3g_{rs}$,

but without transforming $\tau$ [$\delta\tau(\tau) = 0$], this Noether identity is rewritten as

$$
\begin{aligned}
&\delta^{'}_{o\tau}\mathcal{L}_{ADM}(\tau,\vec{\sigma}) = \\
&= L_N(\tau,\vec{\sigma})\delta^{'}_{o\tau}N(\tau,\vec{\sigma}) + L^s_{\vec{N}}(\tau,\vec{\sigma})\delta^{'}_{o\tau}N_s(\tau,\vec{\sigma}) + L^{uv}_g(\tau,\vec{\sigma})\delta^{'}_{o\tau}\, {}^3g_{uv}(\tau,\vec{\sigma}) + \\
&+\partial_\tau\Big[\tilde{\pi}^N(\tau,\vec{\sigma})\delta^{'}_{o\tau}N(\tau,\vec{\sigma}) + \tilde{\pi}^r_{\vec{N}}(\tau,\vec{\sigma})\delta^{'}_{o\tau}N_r(\tau,\vec{\sigma}) + {}^3\tilde{\Pi}^{uv}(\tau,\vec{\sigma})\delta^{'}_{o\tau}\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] + \\
&+\partial_r\Big[\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r N_s(\tau,\vec{\sigma})}\delta^{'}_{o\tau}N_s(\tau,\vec{\sigma}) + \\
&+\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\, {}^3g_{uv}(\tau,\vec{\sigma})} - 2\partial_s\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\Big)\delta^{'}_{o\tau}\, {}^3g_{uv}(\tau,\vec{\sigma})\Big] + \\
&+\partial_r\partial_s\Big(\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_r\partial_s\, {}^3g_{uv}(\tau,\vec{\sigma})}\delta^{'}_{o\tau}\, {}^3g_{uv}(\tau,\vec{\sigma})\Big) \equiv \\
&\equiv -\dot{\beta}(\tau)\Big({}^3\tilde{\Pi}^{rs}\partial_\tau\, {}^3g_{rs} - \mathcal{L}_{ADM}\Big)(\tau,\vec{\sigma}) + {}^3\tilde{\Pi}^{rs}(\tau,\vec{\sigma})\delta^{'}_{o\tau}\, {}^3g_{rs}(\tau,\vec{\sigma}) + \\
&+\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\, {}^3g_{rs}(\tau,\vec{\sigma})}\,\delta^{'}_{o\tau}\, {}^3g_{rs}(\tau,\vec{\sigma}) + \frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_u\, {}^3g_{rs}(\tau,\vec{\sigma})}\delta^{'}_{o\tau}\partial_u\, {}^3g_{rs}(\tau,\vec{\sigma}) + \\
&+\frac{\partial\mathcal{L}_{ADM}(\tau,\vec{\sigma})}{\partial\partial_u\partial_v\, {}^3g_{rs}(\tau,\vec{\sigma})}\delta^{'}_{o\tau}\partial_u\partial_v\, {}^3g_{rs}(\tau,\vec{\sigma}). \qquad \text{(A25)}
\end{aligned}
$$

Since the generator $\int d^3\sigma\alpha(\tau,\vec{\sigma})\tilde{\mathcal{H}}(\tau,\vec{\sigma})$ gives

$\delta_o N(\tau,\vec{\sigma}) = 0$, $\delta_o N_r(\tau,\vec{\sigma}) = 0$, $\delta_o\, {}^3g_{rs}(\tau,\vec{\sigma}) = -\Big(\frac{\alpha}{N}[N_{r|s} + N_{s|r} - \partial_\tau\, {}^3g_{rs}]\Big)(\tau,\vec{\sigma})$,

to get the previous transformations $\delta^{'}_{o\tau}q^i$ one has to put

$\alpha(\tau,\vec{\sigma}) = \beta(\tau)N(\tau,\vec{\sigma})$ with $\beta(\tau)\rightarrow -\delta\tau(\tau)$ for infinitesimal gauge transformations.

Then, these transformations are generated by

$$
\begin{aligned}
G_\tau[\beta(\tau),\dot{\beta}(\tau)] &= \int d^3\sigma\Big(\beta(\tau)[N\tilde{\mathcal{H}} + N_r\tilde{\mathcal{H}}^r] + \dot{\beta}(\tau)[N\tilde{\pi}^N + N_r\tilde{\pi}^r_{\vec{N}}]\Big)(\tau,\vec{\sigma}) = \\
&= G[\beta N, \beta N_r, \dot{\beta}N, \dot{\beta}N_r].
\end{aligned}
\tag{A26}
$$

Therefore, the most general gauge transformation of the ADM action has the generator

$$
\begin{aligned}
G[\beta N, \alpha_r, \lambda_N, \lambda^{\vec{N}}_r] &= \int d^3\sigma[\beta(\tau)N\tilde{\mathcal{H}} + \alpha_r\tilde{\mathcal{H}}^r + \lambda_N\tilde{\pi}^N + \lambda^{\vec{N}}_r\tilde{\pi}^r_{\vec{N}}](\tau,\vec{\sigma}) = \\
&= G_\tau[\beta,\dot{\beta}] + G_2[\alpha_r - \beta N_r] + \\
&+ G_1[\lambda_N - \dot{\beta}(\tau)N - (\alpha^r - \beta(\tau)N^r)\partial_r N, \\
&\quad \lambda^{\vec{N}}_r - \dot{\beta}(\tau)N_r - [\partial_r(\alpha^s - \beta(\tau)N^s) + \delta^s_r(\alpha^u - \beta(\tau)N^u)\partial_u]N_s] = \\
&= G_\tau[\beta,\dot{\beta}] + G_3[\alpha_r - \beta(\tau)N_r, \lambda_N - \dot{\beta}(\tau)N, \lambda^{\vec{N}}_r - \dot{\beta}(\tau)N_r], \\
&\quad \Downarrow \\
\delta_o N(\tau,\vec{\sigma}) &= \lambda_N(\tau,\vec{\sigma}), \\
\delta_o N_r(\tau,\vec{\sigma}) &= \lambda^{\vec{N}}_r(\tau,\vec{\sigma}), \\
\delta_o\, {}^3g_{rs}(\tau,\vec{\sigma}) &= [(\alpha_r - \beta(\tau)N_r)_{|s} + (\alpha_s - \beta(\tau)N_s)_{|r} + \beta(\tau)\partial_\tau\, {}^3g_{rs}](\tau,\vec{\sigma}).
\end{aligned}
\tag{A27}
$$

Maybe there is also a local invariance with arbitrary $\alpha(\tau,\vec{\sigma}) \not= \beta(\tau)N(\tau,\vec{\sigma})$, but we have not found it.

The transformation generated by $G_\tau$ is what remains of the invariance of the Hilbert action under the diffeomorphisms in $M^4$ given by Eq.(31) of I, which have the infinitesimal form

$\delta_o x^\mu = \xi^\mu(x)$

$[0 = \delta_{diff}S_H = \delta_{diff}S_{ADM} + \delta_{diff}\Sigma_{ADM}]$. Since the diffeomorphisms imply

$\delta_o b^\mu_A = \partial_A\delta_o z^\mu(\tau,\vec{\sigma}) = \partial_A\xi^\mu(z(\tau,\vec{\sigma})) = \partial_\rho\xi^\mu(z(\tau,\vec{\sigma}))b^\rho_A(\tau,\vec{\sigma})$,

their effect on the ADM variables in $\Sigma_\tau$-adapted coordinates is the transformations

$\delta_o\, {}^4g_{AB} = \delta_o(b^\mu_A\, {}^4g_{\mu\nu}b^\nu_B) = b^\mu_A b^\nu_B\delta_o\, {}^4g_{\mu\nu} + ({}^4g_{CB}b^\rho_A + {}^4g_{AC}b^\rho_B)b^C_\sigma\partial_\rho\xi^\sigma = -{}^4g_{AB}\partial_\rho\xi^\rho$,

so that from Eq.(6) of I we get

$\delta_o\, {}^3g_{rs} = -{}^3g_{rs}\partial_\rho\xi^\rho$, $\delta_o N^r = 0$, $\delta_o N_r = -N_r\partial_\rho\xi^\rho$, $\delta_o N = -\frac{1}{2}N\partial_\rho\xi^\rho$.

But this is a unique mixture of Eqs.(A23) and of a transformations generated by $\tilde{\pi}^N$. If

we assume the form

$\eta = -\partial_\rho \xi^\rho = \beta(\tau) N$
for the infinitesimal parameter, the associated Hamiltonian generator is

$G_\tau[\beta] + G_1[-\frac{1}{2}\beta N, 0]$.

However, for a generic $\eta = -\partial_\rho \xi^\rho$ we get a 3-conformal infinitesimal transformation $\delta_o\, {}^3g_{rs} = \eta\, {}^3g_{rs}$ enlarged with $\delta_o N = \frac{1}{2}\eta N$, $\delta_o N_r = \eta N_r$. From Appendix B of II [see after Eqs.(B1)] with $\phi = 1 + \frac{1}{4}\eta$ we get

$$\begin{aligned}
&\delta_o \sqrt{\gamma} = \tfrac{3}{2}\eta\sqrt{\gamma},\\
&\delta_o\, {}^3\Gamma^u_{rs} = \tfrac{1}{2}(\delta^u_r \partial_s \eta + \delta^u_s \partial_r \eta - {}^3g_{rs}\, {}^3g^{uv}\partial_v \eta),\\
&\delta_o\, {}^3R = -\eta\, {}^3R - 2\triangle\eta,\\
&\delta_o\, {}^3K_{rs} = \tfrac{1}{2}\eta\, {}^3K_{rs} + \tfrac{{}^3g_{rs}}{2N}(N_u\, {}^3g^{uv}\partial_v\eta - \partial_\tau\eta),
\end{aligned}$$

so that

$$\delta_o \mathcal{L}_{ADM} = -\eta \mathcal{L}_{ADM} + 2\epsilon k \sqrt{\gamma}[N\triangle\eta + {}^3K(N_u\, {}^3g^{uv}\partial_v\eta - \partial_\tau\eta)].$$

Namely, the ADM Lagrangian density is not quasi-invariant under the diffeomorphisms of $M^4$ rewritten in $\Sigma_\tau$-adapted coordinates, but only under the gauge transformations generated by the first class constraints.

See Appendix D for a review of the second Noether theorem in the case of the Hilbert action and for the consequences of its 4-diffeomorphism invariance like the Komar superpotential and the energy-momentum pseudotensors. More work is needed to see whether it is possible to define an ADM superpotential and an associated ADM energy-momentum pseudo-tensor deriving from the second Noether theorem applied to the ADM action.

As said in the Conclusions of II, the consequences of this difference in the invariance properties of the ADM and Hilbert actions (even if they generate the same Einstein's equations) are the following:

i) a "Hamiltonian kinematical gravitational field", defined as an equivalence class of spacetimes modulo the Hamiltonian group of gauge transformations (whose group manifold is not well understood in the large), in general contains many 4-geometries (the elements of $Riem\, M^4 / Diff\, M^4$, which are the standard "kinematical gravitational fields") connected by arbitrary gauge transformations;
ii) a "Hamiltonian Einstein or dynamical gravitational field" is a kinematical one satisfying the Hamilton-Dirac equations (namely all Einstein's equations): it coincides with a standard "Einstein or dynamical gravitational field" (a 4-geometry whose 4-metrics satisfy Einstein's equations), because on the space of solutions of Einstein equations the spacetime diffeomorphisms are solutions of the Jacobi equation associated with Einstein's equations, so that they are dynamical symmetries of Einstein's equations. But this implies that on the space of solutions of Einstein's equations the group manifold of the Hamiltonian gauge transformations is constrained to contain only those gauge transformations which are also

dynamical symmetries of the Hamilton-Dirac equations (and, therefore, also of Einstein's equations). The allowed gauge transformations are the subset of spacetime diffeomorphisms under which the ADM action is quasi-invariant; the other spacetime diffeomorphisms are dynamical symmetries of the equations of motion but not Noether symmetries of the ADM action.

## APPENDIX B: PROPOSALS FOR THE REDUCED PHASE SPACE OF METRIC GRAVITY.

Besides our approach in I and II based on tetrad gravity, the four main proposals to find a copy of the reduced phase space of metric gravity present in the literature are:

1) The (non canonical) one of ADM [4] [see the review in Ref. [5]; in Ref. [33] it is called the 'isotropic radiation gauge']. The canonical variables ${}^3g_{\check r\check s}(\tau,\vec\sigma)$, ${}^3\tilde\Pi^{\check r\check s}(\tau,\vec\sigma)$ are divided (not in a canonical way) in a gauge sector of four pairs ${}^3g_T$, ${}^3\tilde\Pi_T$, ${}^4g_{L\,\check r}$, ${}^3\tilde\Pi^{\check r}_L$, and in a physical sector ${}^3g_{TT\,\check r\check s}$, ${}^3\tilde\Pi^{\check r\check s}_{TT}$, by means of a "flat transverse-traceless" decomposition (see Appendix C of II for the general TT decomposition) of ${}^3g_{\check r\check s}-\delta_{\check r\check s}$ and of ${}^3\tilde\Pi^{\check r\check s}$ in the spirit of the linearized theory of metric gravity. Namely, one uses the following splitting of symmetric tensors [$\triangle=-\vec\partial^2$ is the flat Laplacian; $\triangle^{-1}$ is assumed to exist and to vanish at spatial infinity]

$$f_{rs}=f_{sr}=f_{TT\,rs}+f_{T\,rs}+\partial_rf_{L\,s}+\partial_sf_{L\,r},$$

$$\begin{aligned}
f_{L\,r}&=-\frac{1}{\triangle}(\partial_sf_{rs}+\frac{1}{2\triangle}\partial_r\partial_s\partial_uf_{us}),\\
f_{T\,rs}&=\frac{1}{2}(\delta_{rs}f_T+\frac{1}{\triangle}\partial_r\partial_sf_T),\\
f_T&=f_{rr}+\frac{1}{\triangle}\partial_r\partial_sf_{rs}.
\end{aligned}\tag{B1}$$

The four gauge-fixings, giving a fixation of the coordinates to asymptotic Minkowski rectangular ones, are

$$\begin{aligned}
&{}^3\tilde\Pi_T(\tau,\vec\sigma)\approx 0,\\
&{}^3g_{L\,\check r}(\tau,\vec\sigma)-\sigma^{\check r}\approx 0,
\end{aligned}\tag{B2}$$

and one assumes that the constraints $\tilde{\mathcal H}(\tau,\vec\sigma)\approx 0$, ${}^3\tilde{\mathcal H}^{\check r}(\tau,\vec\sigma)\approx 0$, may be solved in the form

$${}^3g_T-F[{}^3g_{TT},{}^3\tilde\Pi_{TT}]\approx 0\qquad ,\ {}^3\tilde\Pi^{\check u}_L-G^{\check u}[{}^3g_{TT},{}^3\tilde\Pi_{TT}]\approx 0.$$

The non-canonical variables have the Poisson brackets [the Dirac brackets with respect to the gauge-fixings could be evaluated]

$$\begin{aligned}
&\{{}^3g_T(\tau,\vec\sigma),{}^3\tilde\Pi_T(\tau,\vec\sigma^{'})\}=2\delta^3(\vec\sigma,\vec\sigma^{'}),\\
&\{{}^3g_{L\,\check r}(\tau,\vec\sigma),{}^3\tilde\Pi^{\check s}_L(\tau,\vec\sigma^{'})\}=\frac{1}{2\triangle}(\delta_{\check r\check s}+\frac{1}{2\triangle}\partial_{\check r}\partial_{\check s})\delta^3(\vec\sigma,\vec\sigma^{'}),\\
&\{{}^3g_{TT\,\check r\check s}(\tau,\vec\sigma),{}^3\tilde\Pi^{\check u\check v}_{TT}(\tau,\vec\sigma^{'})\}=\delta_{TT}{}^{\check u\check v}_{\check r\check s}\delta^3(\vec\sigma,\vec\sigma^{'}),
\end{aligned}$$

$$\begin{aligned}
\delta_{TT}{}^{\check u\check v}_{\check r\check s}&=\frac{1}{2}[(\delta_{\check r\check v}+\frac{\partial_{\check r}\partial_{\check v}}{\triangle})(\delta_{\check s\check u}+\frac{\partial_{\check s}\partial_{\check u}}{\triangle})+\\
&+(\delta_{\check r\check u}+\frac{\partial_{\check r}\partial_{\check u}}{\triangle})(\delta_{\check s\check v}+\frac{\partial_{\check s}\partial_{\check v}}{\triangle})+(\delta_{\check r\check s}+\frac{\partial_{\check r}\partial_{\check s}}{\triangle})(\delta_{\check u\check v}+\frac{\partial_{\check u}\partial_{\check v}}{\triangle})],
\end{aligned}$$

$$\delta_{TT}{}^{\check r\check s}_{\check r\check s}=2,\quad \delta_{TT}{}^{\check u\check v}_{\check r\check s}=\delta_{TT}{}^{\check r\check s}_{\check u\check v},\quad \delta_{TT}{}^{\check u\check v}_{\check r\check r}=\partial_{\check r}\delta_{TT}{}^{\check u\check v}_{\check r\check s}=0,$$
$$\delta_{TT}{}^{\check u\check v}_{\check r\check s}\delta_{TT}{}^{\check m\check n}_{\check u\check v}=\delta_{TT}{}^{\check m\check n}_{\check r\check s},\quad \delta_{TT}{}^{\check u\check v}_{\check r\check s}f_{TT}{}^{\check r\check s}=f_{TT}{}^{\check u\check v}. \tag{B3}$$

Without a study of the time constancy of the gauge-fixings and, therefore, of the allowed behaviour of the lapse and shift functions, ADM take into account only the ADM energy with asymptotic Minkowski rectangular coordinates $P^\tau_{ADM}=\int_{S^2_{\tau,\infty}}d^2\Sigma_{\check u}(\partial_{\check r}\,{}^3g_{\check u\check r}-\partial_{\check u}\,{}^3g_{\check r\check r})(\tau,\vec\sigma)\approx P^\tau_{ADM}[{}^3g_{TT},{}^3\tilde\Pi_{TT}]$ and assume that the Hamiltonian in the reduced phase space is $P^\tau_{ADM}$ in its volume form $\hat P^\tau_{ADM}$.

2) The one of Dirac [145] [see the review in Ref. [5]; in Ref. [33] it is called "radiation gauge"] of fixing only the one parameter family of spacelike hypersurfaces $\Sigma_\tau$ by the "maximal slicing condition" ${}^3\tilde\Pi(\tau,\vec\sigma)\approx 0$ [i.e. ${}^3K(\tau,\vec\sigma)\approx 0$; as noted in Ref. [33], Lichnerowicz realized that this condition produces an effective anti-focusing condition, which cures the coordinate singularities (created by the focusing of geodesics in $M^4$ normal to $\Sigma_\tau$) of the 'synchronous' reference systems ($N=1$, $N_r=0$); for closed universes it is better to use ${}^3K(\tau,\vec\sigma)\approx const.$]. Here the canonical variables are divided into one gauge pair (see Appendix C of II for the notation)

$$\varphi=log\,\phi^4=\tfrac{2}{3}log(-\mathcal{P}_{\mathcal{T}})=\tfrac{1}{3}log\,\gamma\qquad ,\ {}^3\tilde\Pi=-2\epsilon k\sqrt{\gamma}\,{}^3K=\tfrac{3}{2}\sqrt{\gamma}\mathcal{T},$$

and into five other pairs

$${}^3\sigma_{\check r\check s}=\gamma^{-1/3}\,{}^3g_{\check r\check s}\ [{}^3\sigma=det\,|{}^3\sigma_{\check r\check s}|=1\qquad],\ {}^3\tilde\Pi^{\check r\check s}_B=\gamma^{1/3}\,{}^3\tilde\Pi^{\check r\check s}_A=\gamma^{1/3}({}^3\tilde\Pi^{\check r\check s}-\tfrac{1}{3}{}^3g^{\check r\check s}\,{}^3\tilde\Pi)$$

with Poisson brackets

$$\begin{aligned}
&\{\varphi(\tau,\vec\sigma),{}^3\tilde\Pi(\tau,\vec\sigma^{'})\}=\delta^3(\vec\sigma,\vec\sigma^{'}),\\
&\{{}^3\sigma_{\check r\check s}(\tau,\vec\sigma),{}^3\tilde\Pi^{\check u\check v}_B(\tau,\vec\sigma^{'})\}=\tilde\delta^{\check u\check v}_{\check r\check s}(\vec\sigma)\delta^3(\vec\sigma,\vec\sigma^{'}),\\
&\{{}^3\tilde\Pi^{\check r\check s}_B(\tau,\vec\sigma),{}^3\tilde\Pi^{\check u\check v}_B(\tau,\vec\sigma^{'})\}=\frac{1}{3}[{}^3\tilde\Pi^{\check r\check s}_B\,{}^3\sigma^{\check u\check v}-{}^3\tilde\Pi^{\check u\check v}_B\,{}^3\sigma^{\check r\check s}](\tau,\vec\sigma)\delta^3(\vec\sigma,\vec\sigma^{'}),\\
&\\
&\tilde\delta^{\check u\check v}_{\check r\check s}(\vec\sigma)=\frac{1}{2}(\delta^{\check u}_{\check r}\delta^{\check v}_{\check s}+\delta^{\check v}_{\check r}\delta^{\check u}_{\check s})-\frac{1}{3}[{}^3\sigma_{\check r\check s}\,{}^3\sigma^{\check r\check s}](\tau,\vec\sigma),\\
&\tilde\delta^{\check r\check s}_{\check r\check s}(\vec\sigma)=5,\quad \tilde\delta^{\check r\check s}_{\check u\check v}(\vec\sigma)\tilde\delta^{\check u\check v}_{\check m\check n}(\vec\sigma)=\tilde\delta^{\check r\check s}_{\check m\check n}(\vec\sigma),\\
&\tilde\delta^{\check u\check v}_{\check r\check s}(\vec\sigma)\,{}^3\sigma_{\check u\check v}(\tau,\vec\sigma)=\tilde\delta^{\check u\check v}_{\check r\check s}(\vec\sigma)\,{}^3\sigma^{\check r\check s}(\tau,\vec\sigma)=0,\quad \tilde\delta^{\check u\check v}_{\check r\check s}(\vec\sigma)\,{}^3\tilde\Pi^{\check r\check s}_B(\tau,\vec\sigma)={}^3\tilde\Pi^{\check u\check v}_B(\tau,\vec\sigma).
\end{aligned}\tag{B4}$$

The gauge-fixing ${}^3\tilde\Pi(\tau,\vec\sigma)\approx 0$ implies that the constraint $\tilde{\mathcal{H}}(\tau,\vec\sigma)\approx 0$ has to be solved in $\varphi(\tau,\vec\sigma)$ [see the Lichnerowicz equation in Appendix C of II] and one can evaluate the Dirac brackets. It is assumed that $P^\tau_{ADM}$ is the real Hamiltonian and, by restricting the asymptotic behaviour of $\partial_{\check u}\,{}^3g_{\check r\check s}(\tau,\vec\sigma)=O(r^{-2})$ to $\partial_{\check u}\,{}^3\sigma_{\check r\check s}(\tau,\vec\sigma)=O(r^{-(2+\epsilon)})$ one has

$$P^\tau_{ADM}=-2\int_{S^2_{\tau,\infty}}d^2\Sigma_{\check u}\partial_{\check u}\gamma^{1/3},$$

because the change from rectangular coordinates $\sigma^{\check r}$ to $\bar\sigma^{\check r}\approx_{r\rightarrow\infty}\sigma^{\check r}(1-\frac{M}{16\pi r})$ gives

$$ds^2\rightarrow_{r\rightarrow\infty}\epsilon(1-\tfrac{M}{8\pi r})d\tau^2+(1+\tfrac{M}{8\pi r})\delta_{\check r\check s}d\bar\sigma^{\check r}d\bar\sigma^{\check s}.$$

Since the 3-coordinates are not fixed, one has still the supermomentum constraints and their gauge freedom.

3) This state of affairs was further developed by York [33] (see also Ref. [30]) with the so called "quasi-isotropic" (QI) coordinate conditions, which are not phase space gauge-fixings [in Ref. [33] there is no discussion of supertranslations]. He parametrizes the canonical variables ${}^3g_{\check r\check s}$, ${}^3\tilde\Pi^{\check r\check s}$, as in Eq. (C5) of Appendix C of II using a TT-decomposition and then adds the following quasi-isotropic coordinate conditions [the first one becomes the maximal slicing condition if put =0; the second one is equivalent to $\partial_{\check s}\, {}^3\sigma_{\check r\check s}(\tau,\vec\sigma) = O(r^{-3})$ in asymptotic rectangular coordinates]

$$\begin{aligned}
&\mathcal{T}(\tau,\vec\sigma) = \frac{2}{3\sqrt{\gamma(\tau,\vec\sigma)}}{}^3\tilde\Pi(\tau,\vec\sigma) = -\frac{4}{3}\epsilon k\, {}^3K(\tau,\vec\sigma) = O(r^{-3}),\\
&[k^{\check r}_{(u)}\, {}^3\nabla^{\check s}\, {}^3\psi_{\check r\check s}](\tau,\vec\sigma) = O(r^{-3}),
\end{aligned}\tag{B5}$$

with ${}^3\psi_{\check r\check s} = {}^3\psi_{TT\,\check r\check s} + {}^3\psi_{L\,\check r\check s} = {}^3h_{\check r\check s} - \frac{1}{3}{}^3h\, {}^3f_{\check r\check s}$ $[{}^3h = {}^3f^{\check r\check s}\, {}^3h_{\check r\check s}]$, if one puts ${}^3g_{\check r\check s} = {}^3f_{\check r\check s} + {}^3h_{\check r\check s}$ with ${}^3f_{\check r\check s}$ being a flat asymptotic 3-metric; $k^{\check r}_{(u)}$ are three unit orthogonal translational Killing vectors of ${}^3f_{\check r\check s}$. York assumes $\mathcal{L}_{\vec k_{(u)}}\, {}^3h_{\check r\check s} = 0$ and ${}^3K^{\check r}{}_{\check s} = O(r^{-2})$ [sometimes the tidal forces are restricted by 'curvature conditions' $\mathcal{L}_{\vec k_{(u)}}\mathcal{L}_{\vec k_{(v)}}\, {}^3h^{\check r}{}_{\check s} = O(r^{-3})$ and $\mathcal{L}_{\vec k_{(u)}}\, {}^3K^{\check r}{}_{\check s} = O(r^{-3})$].

After having posed these quasi-isotropic coordinate conditions (replacing the gauge-fixings to the secondary constraints, see the end of Section V of I), York puts the following gauge-fixings on N and $N_{\check r}$ (replacing the time constancy of the gauge-fixings to the secondary constraints, which produce the gauge-fixings to the primary ones, i.e. which give equations for N and $N_{\check r}$)

$$\begin{aligned}
&\partial_\tau\, {}^3K(\tau,\vec\sigma) = 0,\\
&[{}^3\nabla^{\check s}\, {}^3\Sigma_{\check r\check s}](\tau,\vec\sigma) = [{}^3\nabla^{\check s}\gamma^{1/3}\,\partial_\tau\, {}^3\sigma_{\check r\check s}](\tau,\vec\sigma) =\\
&= {}^3\nabla^{\check s}[-2N({}^3K_{\check r\check s} - \frac{1}{3}{}^3g_{\check r\check s}\, {}^3K) + (L\vec N)_{\check r\check s}](\tau,\vec\sigma) = 0,
\end{aligned}\tag{B6}$$

where ${}^3\Sigma$ is called the "distortion tensor" [it measures the change of shape of a small spheroid dragged along $\tau$ from $\Sigma_\tau$ to $\Sigma_{\tau+d\tau}$] and $(L\vec N)_{\check r\check s} = N_{\check r|\check s} + N_{\check s|\check r} - \frac{2}{3}{}^3g_{\check r\check s}N_{\check u|\check u}$ [see Eq.(C2) of Appendix C of II]. The equation $\partial_\tau\, {}^3K(\tau,\vec\sigma) = 0$ becomes an elliptic equation for the lapse function $N(\tau,\vec\sigma)$ by using the Einstein equations for $\partial_\tau\, {}^3g_{\check r\check s}$ and $\partial_\tau\, {}^3K_{\check r\check s}$; in Ref. [33] it is shown that this equation, after having used the superhamiltonian constraint, is $-{}^3\nabla_{\check u}\, {}^3\nabla^{\check u}N + N\, {}^3K_{\check r\check s}\, {}^3K^{\check r\check s} + N^{\check r}\, {}^3\nabla_{\check r}\, {}^3K \circeq 0$. Then, by assuming $N(\tau,\vec\sigma) = 1 + O(r^{-1})$, $N_{\check r}(\tau,\vec\sigma) = O(r^{-1})$, York linearizes these equations with respect to the flat 3-metric ${}^3f_{\check r\check s}$ and obtains the following (not phase space) gauge-fixings

$$\begin{aligned}
&{}^3f^{\check r\check s}\, {}^3K_{\check r\check s}(\tau,\vec\sigma) = 0,\\
&\partial_\tau[{}^3\nabla^{(f)\check s}\, {}^3\psi_{\check r\check s}](\tau,\vec\sigma) = 0.
\end{aligned}\tag{B7}$$

It is shown in Ref. [33] that the ADM 'isotropic radiation gauge' is equivalent to the special case ${}^3f^{\check r\check s}\, {}^3K_{\check r\check s} = {}^3\nabla^{(f)\check s}\, {}^3\psi_{\check r\check s} = 0$, while Dirac's coordinate conditions ${}^3f^{\check r\check s}\, {}^3K_{\check r\check s} = \partial^{\check s}\, {}^3\sigma_{\check r\check s} = 0$ are a non-covariant approximation of the ADM gauge.

Therefore, in some sense, York's construction can be rephrased in phase space as first fixing N and $N_{\check r}$ (i.e. giving first the gauge-fixings to the primary constraints) and then

deducing gauge-fixings for the secondary constraints, a method which is not natural in constraint theory.

In Ref. [30], as already said, it is shown that the quasi-isotropic coordinate conditions kill the supertranslations and define a unique asymptotic Poincaré group.

4) Instead, in Ref. [6] one imposes the following gauge-fixings to the secondary constraints

$$\begin{aligned}
&{}^3\tilde{\Pi}(\tau,\vec{\sigma}) \approx 0,\\
&{}^3g^{\check{r}\check{s}}(\tau,\vec{\sigma})\, {}^3\Gamma^{\check{u}}_{\check{r}\check{s}}(\tau,\vec{\sigma}) \approx 0 \quad or \quad \partial_{\check{r}}[\sqrt{\gamma}\, {}^3g^{\check{r}\check{s}}](\tau,\vec{\sigma}) \approx 0,
\end{aligned} \tag{B8}$$

corresponding to the maximal slicing condition and to the request of using 3-harmonic coordinates. Asking their time constancy $[\partial_\tau\, {}^3\tilde{\Pi} \approx 0,\, \partial_\tau({}^3g^{\check{r}\check{s}}\, {}^3\Gamma^{\check{u}}_{\check{r}\check{s}}) \approx 0]$, one gets a homogeneous system for N and $N_{\check{r}}$

$$\begin{aligned}
&{}^3\nabla_{\check{u}}\, {}^3\nabla^{\check{u}} N + {}^3K^{\check{r}\check{s}}\, {}^3K_{\check{r}\check{s}} N \approx 0,\\
&{}^3\nabla_{\check{u}}\, {}^3\nabla^{\check{u}} N^{\check{r}} + 2\, {}^3K^{\check{r}\check{s}}\, {}^3\nabla_{\check{s}} N - 2N\, {}^3K^{\check{u}\check{v}}\, {}^3\Gamma^{\check{r}}_{\check{u}\check{v}} \approx 0.
\end{aligned} \tag{B9}$$

If $N, N_{\check{r}}$ are requested to be parameters of "proper" gauge transformations $[N = m,\, N_{\check{r}} = m_{\check{r}}]$, then the only solution is $N = N_{\check{r}} = 0$; in the case of improper gauge transformations there is a unique solution of these elliptic equations with both N and $N_{\check{r}}$ growing linearly with $\vec{\sigma}$.

In this way, it is shown in Ref. [6] that one can construct a copy of the reduced phase space (the gauge-fixings intersect all the gauge orbits once modulo the non studied problem of Gribov ambiguity), so that, by going to Dirac brackets, a unique asymptotic Poincaré group is selected and there are no more supertranslations. Then, in Ref. [6] there is a comparison of these results with the SPI formalism in the coordinate-dependent formulation of Ref. [21] with an extension of the results of Ref. [29].

All the formulations agree on the form of the ADM energy-momentum $P^\tau_{ADM}$, $P^{\check{r}}_{ADM}$, which represents the conserved energy-momentum of the $\tau$-slice $\Sigma_\tau$ of the asymptotically flat spacetime $M^4$. It can be shown that $P^{(\mu)}_{ADM}$ is a four-vector under the asymptotic Poincaré group, when this one is uniquely defined.

## APPENDIX C: SPINORS ON $M^4$ AND ON $\Sigma_\tau$.

Let us add some notions about SL(2,C) spinors on $M^4$ and SU(2) spinors on $\Sigma_\tau$ [77,78]. Our pseudoRiemannian spacetime $(M^4, {}^4g_{\mu\nu})$ is assumed to admit a spin [or spinorial or principal SL(2,C)-bundle] structure. Therefore, we can consider an associated bundle over $M^4$ with structure group SL(2,C) and standard fiber a complex 2-dimensional vector space V equipped with a non-degenerate symplectic 2-form $\epsilon$; elements of V are denoted $\xi^{\tilde A}$ [they are SL(2,C) spinor fields] and the symplectic 2-form has components $\epsilon_{\tilde A\tilde B} = -\epsilon_{\tilde B\tilde A}$ with inverse $\epsilon^{\tilde A\tilde B}$ [$\epsilon^{\tilde A\tilde B}\epsilon_{\tilde B\tilde C} = \delta^{\tilde A}_{\tilde B}$]. If $L \in SL(2,C)$, then $\epsilon_{\tilde A\tilde B}L^{\tilde A}{}_{\tilde C}L^{\tilde B}{}_{\tilde D} = \epsilon_{\tilde C\tilde D}$. If $V^*$ is the dual space of V, with elements $\xi_{\tilde A}$, $\epsilon$ gives an isomorphism between it and V, namely $\epsilon$ can be used to lower and raise indices: $\xi^{\tilde A} = \epsilon^{\tilde A\tilde B}\xi_{\tilde B}$, $\xi_{\tilde B} = \xi^{\tilde A}\epsilon_{\tilde A\tilde B}$. The complex conjugate vector space $\bar V$, with elements $\bar\xi^{\tilde A^{'}}$ (the primed spinors), and its dual $\bar V^*$, with elements $\bar\xi_{\tilde A'}$, are also isomorphic (namely $\bar\epsilon$ and its inverse can be used to lower and raise primed indices). The SL(2,C) spinors in V and $\bar V$ correspond to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of SL(2,C) [when representing massive particles, they satisfy the Klein-Gordon equation, while in the massless case they satisfy Weyl spinorial equations]. Given SL(2,C) spinor fields on $M^4$, we can restrict them to SL(2,C) spinor fields over a spacelike hypersurface in $M^4$ like $\Sigma_\tau$. To relate SL(2,C) spinors to 4-tensors over $M^4$ in each point $p \in M^4$, one considers the 4-dimensional real vector space $W_p$ of all objects $\xi^{\tilde A\tilde A^{'}} = \epsilon\bar\xi^{\tilde A\tilde A^{'}}$ [$\epsilon(+---)$ is the signature of $M^4$, not to be confused with the symplectic 2-form], which is equipped with the natural metric $\epsilon_{\tilde A\tilde B}\bar\epsilon_{\tilde A'\tilde B'}$ with signature $\epsilon(+---)$. Therefore, it is natural to identify $W_p$, at each point $p \in M^4$, with the tangent space $T_pM^4$ at that point through an isomorphism $\sigma^\mu_{\tilde A\tilde A'}$ [called SL(2,C) "soldering form"; it is unique and globally defined if $M^4$ admits a spinor structure] with inverse $\sigma_\mu{}^{\tilde A\tilde A^{'}}$: ${}^4V^\mu = \sigma^\mu{}_{\tilde A\tilde A'}\xi^{\tilde A\tilde A^{'}}$. Since tangent vectors are real, one has $\bar\sigma_\mu{}^{\tilde A\tilde A^{'}} = -\sigma_\mu{}^{\tilde A\tilde A^{'}}$. The real spinor $\bar\xi^{\tilde A^{'}}\xi^{\tilde A}$ corresponds to a null vector $t^\mu = \sigma^\mu{}_{\tilde A\tilde A'}\bar\xi^{\tilde A^{'}}\xi^{\tilde A}$, $t^2 = 0$. Then, one has ${}^4g^{\mu\nu} = \sigma^\mu{}_{\tilde A\tilde A'}\sigma^\nu{}_{\tilde B\tilde B'}\epsilon^{\tilde A\tilde B}\bar\epsilon^{\tilde A^{'}\tilde B^{'}}$ and there is a 2-1 homomorphism from the local SL(2,C) transformation group on spinors to the local proper Lorentz group of $(M^4, {}^4g_{\mu\nu})$: $L^\mu{}_\nu = \sigma^\mu{}_{\tilde A\tilde A'}\sigma_\nu{}^{\tilde B\tilde B^{'}}L^{\tilde A}{}_{\tilde B}\bar L^{\tilde A^{'}}{}_{\tilde B'}$. The torsion-free covariant derivative ${}^4\nabla_\mu$ on $M^4$ with ${}^4\nabla_\mu\, {}^4g_{\alpha\beta} = 0$ is uniquely extended to a covariant derivative ${}^4\tilde\nabla_\mu$ acting on spinors with ${}^4\tilde\nabla_\mu\, \epsilon_{\tilde A\tilde B} = 0$ through the requirement to be compatible with the soldering form: ${}^4\tilde\nabla_\mu\, \sigma^\nu{}_{\tilde A\tilde A'} = 0$ [see Ref. [78] for the relation between its curvature tensor and the Riemann tensor of $M^4$].

As shown for instance in Ref. [131] given a "spin basis" $o^{\tilde A}$, $i^{\tilde A}$ [$\epsilon_{\tilde A\tilde B}o^{\tilde a}i^{\tilde B} = 1$] for V, it induces a null tetrad in $M^4$ (like the one of Appendix F):

$$L^\mu = \sigma^\mu{}_{\tilde A\tilde A'}o^{\tilde A}\bar o^{\tilde A^{'}},\ \mathcal{N}^\mu = \sigma^\mu{}_{\tilde A\tilde A'}i^{\tilde A}\bar i^{\tilde A^{'}},\ M^\mu = \sigma^\mu{}_{\tilde A\tilde A'}o^{\tilde A}\bar i^{\tilde A^{'}},\ \bar M^\mu = \sigma^\mu{}_{\tilde A\tilde A'}i^{\tilde A}\bar o^{\tilde A^{'}}.$$

Let now $({}^3\Sigma, {}^3g_{rs})$ be an abstract Riemannian 3-manifold with torsion-free derivative ${}^3\nabla_u$ [${}^3\nabla_u\, {}^3g_{rs} = 0$]. Since 3-manifolds are parallelizable, they always admit a spin [principal SU(2)-bundle] structure. As before consider the associated bundle over ${}^3\Sigma$ with standard fiber the 2-dimensional complex space V with symplectic 2-form $\epsilon_{\tilde A\tilde B}$. One needs an extra structure on each fiber, namely a positive definite Hermitian inner product $< ., . >$, $< \xi, \eta > = \bar\xi^{\tilde A^{'}}G_{\tilde A'\tilde A}\eta^{\tilde A}$, which is equivalent to a positive definite Hermitian metric $G_{\tilde A'\tilde B} = \bar G_{\tilde A'\tilde B}$ with inverse $G^{\tilde A^{'}\tilde B}$ [either $\epsilon_{\tilde A\tilde B}$ or $G_{\tilde A'\tilde B}$ is assumed so normalized to get $\bar\epsilon^{\tilde A^{'}\tilde B^{'}}G_{\tilde A'\tilde A}G_{\tilde B'\tilde B} =$

$\epsilon_{\tilde{A}\tilde{B}}$, so that $G_{\tilde{A}'\tilde{A}}G^{\tilde{A}'\tilde{B}} = \delta^{\tilde{B}}_{\tilde{A}}$]. This metric allows to convert the primed indices to the unprimed ones, so that we can restrict ourselves to unprimed SU(2) spinors. The SU(2) transformations are the ones that preserve both $\epsilon$ and $G$ structure [the Hermitian conjugate of a transformation $U^{\tilde{A}}{}_{\tilde{B}}$ is $(U^\dagger)^{\tilde{C}}{}_{\tilde{D}} = G^{\tilde{B}'\tilde{C}}\bar{U}^{\tilde{A}'}{}_{\tilde{B}'}G_{\tilde{A}'\tilde{D}}$]. The tangent space $T_p\, {}^3\Sigma$ at $p \in {}^3\Sigma$ is always globally isomorphic, through the SU(2) "soldering form" $\sigma^r{}_{\tilde{A}}{}^{\tilde{B}}$, to the 3-dimensional real vector space H of all objects at $p \in {}^3\Sigma$ of the form $\alpha^{\tilde{A}}{}_{\tilde{B}}$ satisfying $\alpha^{\tilde{A}}{}_{\tilde{A}} = 0$, $(\alpha^\dagger)^{\tilde{A}}{}_{\tilde{B}} = -\alpha^{\tilde{A}}{}_{\tilde{B}}$ [H is equipped with a natural positive definite metric $(\alpha, \beta) = -\alpha^{\tilde{A}}{}_{\tilde{B}}\beta^{\tilde{B}}{}_{\tilde{A}}$, so that $\sigma^r{}_{\tilde{A}}{}^{\tilde{A}} = 0$, $(\sigma^r{}_{\tilde{A}}{}^{\tilde{B}})^\dagger = -\sigma^r{}_{\tilde{A}}{}^{\tilde{B}}$]. One has ${}^3g^{rs} = -\sigma^r{}_{\tilde{A}}{}^{\tilde{B}}\sigma^s{}_{\tilde{B}}{}^{\tilde{A}} = -Tr(\sigma^r\sigma^s)$. The objects $\psi^{\tilde{A}}$ are SU(2) spinor fields on $({}^3\Sigma, {}^3g_{rs})$ and SU(2) transformations on spinors, $U^{\tilde{A}}{}_{\tilde{B}}$, are tied to SO(3) transformations on the tangent spaces of ${}^3\Sigma$ by a 2-1 homomorphism $U^r{}_s = \sigma^{r\tilde{A}}{}_{\tilde{B}}U^{\tilde{B}}{}_{\tilde{C}}\sigma^{s\tilde{C}}{}_{\tilde{D}}(U^\dagger)^{\tilde{D}}{}_{\tilde{A}}$. There is a unique extension of ${}^3\nabla_u$ to a covariant derivative ${}^3\tilde{\nabla}_u$ acting on SU(2) spinors such that ${}^3\tilde{\nabla}_u\sigma^r{}_{\tilde{A}}{}^{\tilde{B}} = {}^3\tilde{\nabla}_u\, \epsilon_{\tilde{A}\tilde{B}} = {}^3\tilde{\nabla}_u\, G_{\tilde{A}'\tilde{B}} = 0$.

Let us now consider the Riemannian spacelike hypersurface $(\Sigma_\tau, {}^3g_{rs})$ embedded in the pseudo-Riemannian spacetime $M^4$ and let us define SU(2) spinors on $\Sigma_\tau$ starting from SL(2,C) spinors on $M^4$, just as spatial 3-tensors on $\Sigma_\tau$ are identified with 4-tensors on $M^4$ which are tangent to $\Sigma_\tau$. As at each point of $\Sigma_\tau$ one can identify SO(3) transformations with proper Lorentz transformations which preserve the future-directed unit normal $l^\mu(\tau, \vec{\sigma})$ to $\Sigma_\tau$, similarly SU(2) transformations can be identified with SL(2,C) transformations preserving $l^{\tilde{A}\tilde{A}'} = l^\mu\sigma_\mu{}^{\tilde{A}\tilde{A}'}$ [one has $l_{\tilde{A}\tilde{A}'} = l_\mu\sigma^\mu{}_{\tilde{A}\tilde{A}'}$ and $l_\mu l^\mu = \epsilon$ becomes $l^{\tilde{A}\tilde{A}'}l_{\tilde{B}\tilde{A}'} = \frac{\epsilon}{2}\delta^{\tilde{A}}_{\tilde{B}}$], where $\sigma_\mu{}^{\tilde{A}\tilde{A}'}$ is the SL(2,C) soldering form. To reduce the structure group SL(2,C) of the spin bundle to SU(2), one needs the extra structure of a positive definite Hermitian inner product or, equivalently, of a positive definite Hermitian metric

$$G_{\tilde{A}'\tilde{A}} = \sqrt{2} l_{\tilde{A}\tilde{A}'}$$

[hermiticity follows from $\bar{\sigma}_\mu^{\tilde{B}\tilde{A}'} = -\sigma_\mu{}^{\tilde{A}\tilde{B}'}$]. The SL(2,C) transformations preserving $l_{\tilde{A}\tilde{A}'}$ can be identified with SU(2) transformations. The horizontal subspace of the associated bundle over $M^4$ with standard fiber V with respect to the normal $l_{\tilde{A}\tilde{A}'}$ consists of all the elements $\alpha^{\tilde{A}\tilde{A}'}$ of V satisfying $\alpha^{\tilde{A}\tilde{A}'}l_{\tilde{A}\tilde{A}'} = 0$. This is a 3-dimensional real vector space [with a positive definite metric $\epsilon_{\tilde{A}\tilde{B}}\bar{\epsilon}_{\tilde{A}'\tilde{B}'} - \epsilon\, l_{\tilde{A}\tilde{A}'}l_{\tilde{B}\tilde{B}'} = \epsilon_{\tilde{A}\tilde{B}}\bar{\epsilon}_{\tilde{A}'\tilde{B}'} + \frac{\epsilon}{2}G_{\tilde{A}'\tilde{A}}G_{\tilde{B}'\tilde{B}}$ induced by the metric $\epsilon_{\tilde{A}\tilde{B}}\bar{\epsilon}_{\tilde{A}'\tilde{B}'}$ on V] isometric to the space H used in the theory of SU(2) spinors, through the isometry

$$\alpha^{\tilde{A}\tilde{A}'} \mapsto \alpha^{\tilde{A}}{}_{\tilde{B}} = \alpha^{\tilde{A}\tilde{A}'}G_{\tilde{A}'\tilde{B}}$$

[the metric in H is $G^{\tilde{A}'\tilde{C}}G^{\tilde{B}'\tilde{D}}(\epsilon_{\tilde{A}\tilde{B}}\bar{\epsilon}_{\tilde{A}'\tilde{B}'} + \frac{\epsilon}{2}G_{\tilde{A}'\tilde{A}}G_{\tilde{B}'\tilde{B}}) = \epsilon^{\tilde{C}\tilde{D}}\epsilon_{\tilde{A}\tilde{B}} + \frac{\epsilon}{2}\delta^{\tilde{C}}_{\tilde{A}}\delta^{\tilde{D}}_{\tilde{B}}$]. The SU(2) soldering form is

$$\sigma_\mu{}^{\tilde{A}}{}_{\tilde{B}} = {}^3g_\mu{}^\nu\sigma_\nu^{\tilde{A}\tilde{A}'}G_{\tilde{A}'\tilde{B}}$$

[${}^3g_{\mu\nu} = -\epsilon\, {}^3h_{\mu\nu} = -\epsilon({}^4g_{\mu\nu} - \epsilon l_\mu l_\nu)$ so that ${}^3g_{\tilde{A}\tilde{C}\tilde{B}\tilde{D}} = -\frac{1}{2}(\epsilon_{\tilde{A}\tilde{D}}\epsilon_{\tilde{B}\tilde{C}} + \epsilon_{\tilde{A}\tilde{B}}\epsilon_{\tilde{D}\tilde{C}})$]. Since the spinor field ${}^4\alpha_{\tilde{A}\tilde{A}'}$ on $M^4$ is "spatial" if

${}^4\alpha_{\tilde A\tilde A'} l^{\tilde A\tilde A'} = 0,$

i.e. if ${}^4\alpha_{\tilde A\tilde B} = {}^4\alpha_{\tilde A\tilde A'} l^{\tilde A'}{}_{\tilde B} = {}^4\alpha_{\tilde B\tilde A} = {}^4\alpha_{(\tilde A\tilde B)}$ [$(\tilde A\tilde B)$ means symmetrization], each spatial 3-tensor ${}^3T^{\mu ...\nu}_{\alpha ...\beta}$ defines a spatial unprimed spinor field ${}^3T^{(\tilde A_1\tilde A_2)...(\tilde B_1\tilde B_2)}_{(\tilde C_1\tilde C_2)...(\tilde D_1\tilde D_2)}$. If $\xi_{\tilde A} \mapsto \xi^{\dagger}_{\tilde A} = \bar\xi_{\tilde A'} l^{\tilde A'}{}_{\tilde A}$ is the conjugation map, a "spatial spinor" field represents a "real" spatial 3-tensor field if and only if $({}^3T^{(\tilde A_1\tilde A_2)...(\tilde B_1\tilde B_2)}_{(\tilde C_1\tilde C_2)...(\tilde D_1\tilde D_2)})^{\dagger} = (-)^k\, {}^3T^{(\tilde A_1\tilde A_2)...(\tilde B_1\tilde B_2)}_{(\tilde C_1\tilde C_2)...(\tilde D_1\tilde D_2)}$ with 2k being the number of indices.

The torsion-free covariant derivative ${}^4\nabla_\mu$ [${}^4\nabla_\mu\, {}^4g_{\alpha\beta} = 0$] on $M^4$ goes down to the torsion-free ${}^3\nabla_\mu$ [${}^3\nabla_\mu\, {}^3g_{\alpha\beta} = 0$] acting on spatial 3-tensors on $\Sigma_\tau$ [see after Eq.(8) of I]. They are extended to ${}^4\tilde\nabla_\mu$ [or ${}^4\tilde\nabla_{\tilde A\tilde A'} = \sigma^\mu_{\tilde A\tilde A'}\, {}^4\tilde\nabla_\mu$; ${}^4\tilde\nabla_\mu \epsilon_{\tilde A\tilde B} = {}^4\tilde\nabla_\mu \sigma^\nu{}_{\tilde A\tilde A'} = 0$] acting on SL(2,C) spinors on $M^4$ and to ${}^3\tilde\nabla_{\tilde A\tilde B} = \sqrt{2} l_{(\tilde A}{}^{\tilde A'}\, {}^4\nabla_{\tilde B)\tilde A'}$ (the torsion-free Levi-Civita connection as we have seen) acting on spatial spinors on $\Sigma_\tau$ [${}^3\tilde\nabla_u \sigma^r{}_{\tilde A}{}^{\tilde B} = 0$ and ${}^3\tilde\nabla_u(\epsilon^{\tilde C\tilde D}\epsilon_{\tilde A\tilde B} + \frac{\epsilon}{2}\delta^{\tilde C}_{\tilde A}\delta^{\tilde D}_{\tilde B}) = 0$; it is the Levi-Civita connection of ${}^3g$]. One has the splitting

$${}^4\tilde\nabla_{\tilde A\tilde B} = l^{\tilde A'}{}_{\tilde B}\, {}^4\tilde\nabla_{\tilde A\tilde A'} = \frac{1}{2}\epsilon_{\tilde A\tilde B} T + l_{(\tilde B}{}^{\tilde A'}\, {}^4\tilde\nabla_{\tilde A)\tilde A'}.$$

The first term $T = l^{\tilde C\tilde D'}\, {}^4\tilde\nabla_{\tilde C\tilde D'}$ represents the "time derivative". The second term, which depends only on the intrinsic geometry of $\Sigma_\tau$, represents the "spatial" derivative "only" when $\Sigma_\tau$ has zero extrinsic curvature ${}^3K_{\mu\nu} = 0$.

The true "spatial" derivative , called the "Sen connection", is given by real operators ${}^3\mathcal{D}_{\tilde A\tilde B} = {}^3\mathcal{D}_{\tilde B\tilde A}$ [acting on real spinor fields they produce real spinor fields] which are not only the pullback to $\Sigma_\tau$ of ${}^4\nabla_\mu$ but also an extension [depending also from the extrinsic geometry of $\Sigma_\tau$] of ${}^3\nabla_\mu$ from spatial tensors to SU(2) spinors [the Sen connection is torsion-free, satisfies ${}^3\mathcal{D}_{\tilde A\tilde B}\, {}^3g_{\tilde C\tilde D\tilde E\tilde F} = 0$ but is not the Levi-Civita connection of ${}^3g$]. On scalars one has ${}^3\mathcal{D}_{\tilde A\tilde B}\phi = {}^3\tilde\nabla_{\tilde A\tilde B}\phi$. Instead its action on SU(2) spinors is

$${}^3\mathcal{D}_{\tilde A\tilde B}\, \psi_{\tilde C} = {}^3\tilde\nabla_{\tilde A\tilde B}\, \psi_{\tilde C} + \frac{1}{\sqrt{2}}\, {}^3K_{\tilde A\tilde B\tilde C}{}^{\tilde D}\, \psi_{\tilde D}, \tag{C1}$$

so that one has

$${}^3\mathcal{D}_{\tilde A\tilde B}\, \psi^{\tilde B} = {}^3\tilde\nabla_{\tilde A\tilde B}\psi^{\tilde B} + \frac{1}{2\sqrt{2}}\, {}^3K\psi_{\tilde A}. \tag{C2}$$

## APPENDIX D: KOMAR SUPERPOTENTIAL AND ENERGY-MOMENTUM PSEUDOTENSORS FROM THE HILBERT ACTION.

For the sake of completeness let us add the standard derivation of the Noether identities, the determination of the Komar superpotential and of the energy-momentum pseudotensors in metric gravity starting from the Hilbert action and from its invariance under 4-diffeomorphisms, since it is in this way that one usually defines the weak Poincaré charges (even if with open problems for the ADM angular momentum).

The invariance under $Diff\, M^4$ of the Hilbert action may be used to set the 4-metric tensor equal to the Minkowski 4-metric and the affine connection to zero at any point of $M^4$. Indeed, these conditions may be satisfied along an arbitrary geodesic of $M^4$. It is in this way that the principle of equivalence, the equality of inertial and gravitational mass, is described in general relativity: to first order in their separation, all bodies moving on parallel geodesics move at the same rate. Just this property is also responsible for the inability to define a "local energy density" for the gravitational field. Minkowski spacetime describes a spacetime with no gravitational field, so that its energy density must be zero. But a general spacetime can be made to appear Minkowskian along an arbitrary geodesic. As a result, any non-tensorial (even if covariant) "energy density" can be made to be zero along an arbitrary geodesic and, therefore, has no invariant meaning. To define a tensorial quantity requires the introduction of an auxiliary vector field $\xi^\mu$, which is an element of arbitrariness. It follows that only the global energy-momentum and angular momentum may be given a meaning in general relativity.

Conservation laws in general relativity were first formulated by Einstein in 1916 [146] [Noether's theorems appeared in 1918], who found a canonical pseudo-tensor of the gravitational field homogeneous quadratic in the first derivatives of the metric tensor. Due to its non-tensoriality, the local energy density does not have a covariant significance and was criticized by Schroedinger [147] [he found a coordinate system in which all the components of the pseudo-tensor vanished for the Schwarzschild metric outside the Schwarzschild radius]. Bauer [148] showed that simply by transforming the description of flat space from Cartesian coordinates to spherical coordinates an apparent nonzero "energy density" results which yields an infinite total "energy". This criticism was answered only when Einstein [149,150] showed that the total energy and momentum, the only meaningful quantities, are constants of the motion and transform as a "free-vector" [an affine tensor: a set of quantities which are not defined at a particular point in space] under linear coordinate transformations. This is the so-called Einstein [149]- Klein [151] theorem: it assumes the existence of "asymptotically flat" coordinates such that [152] ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + O(r^{-1})$, $\partial_\alpha\, {}^4g_{\mu\nu} = O(r^{-2})$ [$r$ is the distance measured along geodesics from a point on a spacelike asymptotically flat hypersurface] and, by writing the Schwarzschild line element in coordinates which are Cartesian at infinity, one gets $p^o = m$, $p^i = 0$. Einstein showed that under certain conditions (essentially, no radiation) the total energy and momentum in a closed domain of space (outside one uses Minkowski coordinates) is independent of the choice of the coordinates within the domain [this was called the "flux theorem" by Pauli [153]; see also Refs. [154–156] based on the work of P.von Freud [157], who was the first to find a superpotential for the Einstein pseudo-tensor]. Trautman [158] added conditions for extending the Einstein-Klein results from the case of asymptotically flat isolated non-radiating systems to that of radiating systems [see also the

discussion in Ref. [159], where a background Minkowski metric is used to covariantize the treatment].

The pseudo-tensor was named a "complex" by Lorentz [160], who gave a different, non satisfactory, definition of energy and momentum of the gravitational field. In order to discuss angular momentum, a symmetric energy-momentum tensor is desirable [153], although not necessary [161]. The Einstein canonical pseudo-tensor has mixed indices, and raising one with the metric tensor does not yield a symmetric quantity. Landau and Lifshitz [62] succeeded in constructing a symmetric pseudo-tensor [see Ref. [159] for the associated angular momentum tensor in presence of a flat background], but the associated total energy and momentum transform as a vector density rather than as a vector as in the case of particles. Bergmann [162] started the investigation on the local invariances of the action in general relativity (second Noether theorem) and of their Hamiltonian generators. In the study of the relationship between Landau-Lifshitz and Einstein canonical pseudotensors, Goldberg [163] discovered a whole family of mixed and symmetric pseudotensors with different weights. All the mixed tensors have the same physical content (total energy and momentum), whereas the symmetric ones are all different in their physical content. Of the symmetric quantities, only the Landau-Lifshitz pseudo-tensor has the same total energy and momentum as the Einstein canonical one (but has the wrong transformation properties).

Komar [164], trying to generalize an earlier theory due to Møller [165], looked for a superpotential depending on a vector field $\xi^\mu$, such that, when $\xi^\mu$ is the timelike Killing vector of the Schwarzschild solution, the constant of the motion is the mass. In this way he gets a Noether (weak) conservation law $t^\nu{}_{,\nu} \circeq 0$ in which $t^\nu$ is a tensor. However, this coordinate-free expression depends on the choice of the vector field $\xi^\mu$. In asymptotically flat spacetimes, one takes for $\xi^\mu$ asymptotic Killing vectors to flat spacetime [it works for translations, but there are problems with rotations [166]].

Møller [167] proposed a theory designed to provide a definition of energy invariant under time independent spatial transformations without any restriction on the asymptotic form of the metric. Møller's theory is based on the Hilbert variational principle with the 4-metric reexpressed in terms of orthogonal tetrads. Due to the extra 6 gauge degrees of freedom (there are constraints generating the local Lorentz transformations in the tangent planes), he proposed 6 gauge-fixings. The main point is that, when one assumes that the universe is asymptotically flat [instead of a spatially closed (without boundaries) one], one is introducing an absolute family of privileged observers, namely those whose associated tetrads tend to Minkowski tetrads at space infinity [against the philosophy of Mach's principle, which, to avoid absolute motions, requires a spatially closed universe; see Section IV]. Then Møller [168] applied his theory to the axis-symmetric solution and found that the total energy agrees with the expression of the Bondi mass. Møller's theory gives no information concerning the linear momentum of the system.

contrastedWhile for regular Lagrangian systems proper conservation laws are related to its invariances under global symmetries according to the first Noether theorem, with singular Lagrangians one has improper [either weak (i.e. implying the use of the equation of motion) or strong (i.e. independent from the equations of motion)] conservation laws [169,170], which are hidden in the Noether identities implied by local symmetries (either inner gauge groups or diffeomorphisms of time and/or space and/or spacetime) according to the second Noether theorem [see the original Refs. [155,162,163,171] and Refs. [35] b), c), d) for a systematic

and complete treatment for singular Lagrangians depending upon the first derivatives of the fields]. In general relativity [170] the local symmetry transformations of the theory are the diffeomorphisms of the spacetime $M^4$.

In Refs. [172] there is a complete bibliography on the modern variational techniques based on jet bundles for treating Noether's theorems also for Lagrangians depending on higher derivatives of the fields. This is the Lagrangian approach to be with either the use of the equations of motion and Bianchi identities or the non-covariant Hamiltonian formalism using momentum mapping methods in symplectic or presymplectic manifolds. In particular, in the papers of Refs. [172] one makes use of the formulation based on the globally defined Poincaré-Cartan one-form (uniquely defined for first and second order Lagrangians) applied to general relativity.

Let $\mathcal{L}(x, \phi_A, \phi_{A,\mu}, \phi_{A,\mu\nu}) \stackrel{def}{=} \mathcal{L}(x; \phi(x))$ $[S = \int d^4x\, \mathcal{L}]$ be a singular Lagrangian density depending on a set of fields $\phi_A(x)$ [A=1..N; for Einstein general relativity $\phi_A = {}^4g_{\mu\nu}$ and $\mathcal{L} = \mathcal{L}_H$, not explicitly depending on $x^\mu$, and $S = S_H = \int d^4x\, \mathcal{L}_H$ of Eq.(22) of I] and their first and second derivatives (in Einstein general relativity the dependence on the second derivatives is linear, due to Eq.(3) of I, and can be reabsorbed in a 4-divergence: $\mathcal{L}_H = \mathcal{L}_E + \frac{c^3}{16\pi G}\partial_\lambda[\sqrt{{}^4g}({}^4g^{\lambda\alpha}\, {}^4g_{\mu\nu}\partial_\alpha\, {}^4g^{\mu\nu} - \partial_\mu\, {}^4g^{\lambda\mu})]$ and this defines the (not general covariant) Einstein action of Eq.(24) of I ). The Euler-Lagrange equations are

$$L^A(x, \phi(x)) = \frac{\delta S}{\delta \phi_A(x)} = \frac{\partial \mathcal{L}}{\partial \phi_A}(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A}(x) + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_A}(x) \stackrel{\circ}{=} 0.$$

A given physical situation can be described in different frames of reference (systems of coordinates) and by means of different sets of variables $\phi$, or, for short, in different gauges. The class of finite gauge transformations to be considered is given by

$$x'^{\mu} = X^\mu(x),\; \phi'_A(x') = Y_A(x; \phi)$$

[for metric general relativity ${}^4g'_{\mu\nu}(x'(x)) = \frac{\partial x^\alpha}{\partial x'^{\mu}} \frac{\partial x^\beta}{\partial x'^{\nu}}\, {}^4g_{\alpha\beta}(x)$ ]. It contains Lorentz and general coordinate transformations (diffeomorphisms) and symmetry transformations of classical mechanics as well as electromagnetic gauge transformations.

Given $S = \int_U d^4x\, \mathcal{L}(x; \phi(x))$, a sufficient condition for the new equations of motion for $\phi'_A(x')$ to be equivalent to the old ones is that the new action $S'$ is given by

$$S' = \int_{U'} d^4x'\, \mathcal{L}'(x'; \phi'(x')) = \int_U d^4x\, [\mathcal{L}(x; \phi(x)) - \partial_\mu Q^\mu(x; \phi(x))]$$

for some 4 functions $Q^\mu$. Here, $\mathcal{L}'$ is the new Lagrangian density and $U'$ denotes the image of U by the transformation $x^\mu \mapsto x'^{\mu} = X^\mu(x)$. The functions $Q^\mu$ are arbitrary except for the condition that they must vanish whenever the $\phi$'s and their derivatives vanish. This condition on $S'$ does not ensure numerical invariance of the action ($S \neq S'$), but one has $\delta S = \delta S'$ provided the $\delta\phi$'s vanish with their derivatives on $\partial U$, the boundary of U. From this it follows that the new Lagrangian density $\mathcal{L}'$ is not uniquely determined by $\mathcal{L}$, and, generally, there is an arbitrariness in the choice of the Lagrangian. The Lagrangian density $\mathcal{L}'$ as function of its arguments is different from $\mathcal{L}$. The corresponding new equations of motion $L'^{A}(x; \phi(x)) = \delta S'/\delta\phi'_A(x) \stackrel{\circ}{=} 0$ will differ in form from the old ones.

But let us assume that the gauge transformation $x'^{\mu} = X^\mu(x)$, $\phi'_A(x') + Y_A(x; \phi)$, is a

"symmetry transformation": this means that it leaves the form of the equations of motion unaltered, namely $L^{'A}(x;\phi(x)) = L^A(x;\phi(x))$. It will be so if, for a certain choice of the functions $Q^\mu$, the form of the Lagrangian density is not changed: $\mathcal{L}^{'}(x;\phi(x)) = \mathcal{L}(x;\phi(x))$.

Let us now assume that these gauge symmetries form a continuous group. Then, they can be characterized by the infinitesimal transformations

$$x^{'\mu} = x^\mu + \delta x^\mu(x),\ \phi^{'}_A(x^{'}) = \phi_A(x) + \delta\phi_A(x) = \phi_A(x) + \delta_o\phi_A(x) + \delta x^\mu(x)\partial_\mu\phi_A(x)$$

with $\delta_o\phi_A(x) = \phi^{'}_A(x) - \phi_A(x)$ so that $\mathcal{L}(x;\phi^{'}(x)) - \mathcal{L}(x;\phi(x)) = \delta_o\mathcal{L}$ [$\delta_o$ commutes with differentiation ]. Then one gets

$$\begin{aligned} &d^4x^{'}\mathcal{L}^{'}(x^{'};\phi^{'}(x^{'})) - d^4x\mathcal{L}(x;\phi(x)) = d^4x[(1+\partial_\mu\delta x^\mu)\mathcal{L}(x^{'};\phi^{'}(x^{'})) - \mathcal{L}(x;\phi(x))] = \\ &= d^4x[\mathcal{L}(x;\phi(x))\partial_\mu\delta x^\mu + \delta\mathcal{L}] = d^4x[\mathcal{L}\partial_\mu\delta x^\mu + \delta_o\mathcal{L} + \delta x^\mu\mathcal{L}] \equiv -d^4x\partial_\mu Q^\mu_{INF}, \\ &\Rightarrow \delta\mathcal{L} + \mathcal{L}\partial_\mu\delta x^\mu = \delta_o\mathcal{L} + \partial_\mu(\mathcal{L}\delta x^\mu) \equiv \partial_\mu F^\mu, \end{aligned} \tag{D1}$$

where $F^\mu = -Q^\mu_{INF}$ denotes the functions $Q^\mu$ corresponding to the infinitesimal transformations. This is the statement of "quasi-invariance", which becomes "invariance" when $F^\mu \equiv 0$. Since

$$\delta_o\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_A}\delta_o\phi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\delta_o\partial_\mu\phi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A}\delta_o\partial_\mu\partial_\nu\phi_A = L^A\delta_o\phi_A + \partial_\mu[(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A})\delta_o\phi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A}\delta_o\partial_\nu\phi_A],$$

one gets the Noether identity

$$\partial_\mu\theta^\mu_{(W)} \stackrel{def}{=} \partial_\mu[(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A})\delta_o\phi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A}\delta_o\partial_\nu\phi_A + \mathcal{L}\delta x^\mu - F^\mu] \equiv -\delta_o\phi_A L^A \,\hat{=}\, 0. \tag{D2}$$

If the theory is quasi-invariant under a general group $G_{\infty q}$, that is to say under transformations which depend upon q arbitrary functions $\epsilon^a(x)$, a=1,..,q. of the $x^\mu$'s [in metric general relativity one has the group $G_{\infty 4}$ of diffeomorphisms with arbitrary functions $\epsilon^\mu(x)$], assumed for simplicity of the form

$$\delta x^\mu = \epsilon^a(x)\xi^\mu_a(x),\ \delta_o\phi_A(x) = \epsilon^a(x)\eta_{Aa}(x,\phi,\partial_\mu\phi) + \partial_\nu\epsilon^a(x)\eta^\nu_{Aa}(x,\phi,\partial_\mu\phi),$$

so that

$$F^\mu = \epsilon^a F^\mu_a + \partial_\nu\epsilon^a F^{\mu\nu}_a,$$

then one gets the following Noether identities from the vanishing of the coefficients of the arbitrary functions $\partial_\mu\partial_\nu\partial_\rho\epsilon^a$, $\partial_\mu\partial_\rho\epsilon^a$, $\partial_\rho\epsilon^a$, $\epsilon^a$ [indices inside round brackets are completely symmetrized $t^{(\mu\nu)} = \frac{1}{2}(t^{\mu\nu} + t^{\nu\mu})$]

$$\partial_\mu\theta^\mu_{(W)}[\epsilon^a] = \partial_\mu[\epsilon^a t^\mu_a + \partial_\rho\epsilon^a t^{\mu\rho}_a + \partial_\nu\partial_\rho\epsilon^a\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A}\eta^\rho_{Aa}] \equiv$$

$$\equiv -(\epsilon^a \eta_{Aa} + \partial_\rho \epsilon^a \eta^\rho_{Aa}) L^A \stackrel{\circ}{=} 0,$$

$$\frac{\partial \mathcal{L}}{\partial_{(\mu}\partial_\nu \phi_A}\, \eta^{\rho)}_{Aa} \equiv 0,$$

$$\partial_\nu \Big( \frac{\partial \mathcal{L}}{\partial \partial_\nu \partial_{(\mu}\phi_A} \eta^{\rho)}_{Aa} \Big) \equiv -t_a^{(\mu\rho)} \stackrel{def}{=}$$

$$\stackrel{def}{=} -\Big[\Big( \frac{\partial \mathcal{L}}{\partial \partial_{(\mu}\phi_A} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \partial_{(\mu}\phi_A} \Big) \eta^{\rho)}_{Aa} + \frac{\partial \mathcal{L}}{\partial \partial_\nu \partial_{(\mu}\phi_A} (\delta^{\rho)}_\nu \eta_{Aa} + \partial_\nu \eta^{\rho)}_{Aa}) - F_a^{(\mu\rho)}\Big],$$

$$\partial_\mu t_a^{\mu\rho} + t^\rho_a \stackrel{def}{=} \partial_\mu \Big[\Big( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_A} \Big) \eta^\rho_{Aa} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_A} (\delta^\rho_\nu \eta_{Aa} + \partial_\nu \eta^\rho_{Aa}) - F_a^{\mu\rho}\Big] +$$
$$+ \Big[\Big( \frac{\partial \mathcal{L}}{\partial \partial_\rho \phi_A} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\rho \partial_\nu \phi_A} \Big) \eta_{Aa} + \frac{\partial \mathcal{L}}{\partial \partial_\rho \partial_\nu \phi_A} \partial_\nu \eta^\rho_{Aa} + \mathcal{L} \xi^\rho_a - F^\rho_a\Big] \equiv -\eta^\rho_{Aa} L^A \stackrel{\circ}{=} 0,$$

$$\partial_\mu t^\mu_a \stackrel{def}{=} \partial_\mu \Big[\Big( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_A} \Big) \eta_{Aa} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_A} \partial_\nu \eta_{Aa} + \mathcal{L} \xi^\mu_a - F^\mu_a\Big] \equiv -\eta_{Aa} L^A \stackrel{\circ}{=} 0. \qquad \text{(D3)}$$

These Noether identities imply the "contracted Bianchi identities"

$$\eta_{Aa} L^A - \partial_\rho (\eta^\rho_{Aa} L^A) \equiv 0, \qquad \text{(D4)}$$

and, since $\partial_\mu \partial_\rho t_a^{\mu\rho} = \partial_\mu \partial_\rho t_a^{(\mu\rho)} \equiv \partial_\rho \partial_\mu \partial_\nu \big( \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_{(\nu}\phi_A} \eta^{\rho)}_{Aa} \big) \equiv \partial_\rho \partial_\mu \partial_\nu \big( \frac{\partial \mathcal{L}}{\partial \partial_{(\mu}\partial_\nu \phi_A} \eta^{\rho)}_{Aa} \big) \equiv 0$, one also gets the strong (i.e. independent from the equations of motion) conservation laws

$$\partial_\mu \theta^\mu_{(S)a} \stackrel{def}{=} \partial_\mu (t^\mu_a + \eta^\mu_{Aa} L^A) \equiv 0. \qquad \text{(D5)}$$

The original Noether identities together with the contracted Bianchi identities allow to get

$$\partial_\mu \theta^\mu_{(W)}[\epsilon^a] \equiv -\delta_o \phi_A L^A = -(\epsilon^a \eta_{Aa} + \partial_\rho \epsilon^a \eta^\rho_{Aa}) L^A \equiv -[\partial_\rho \epsilon^a \eta^\rho_{Aa} L^A + \epsilon^a \partial_\rho (\eta^\rho_{Aa} L^A)] = -\partial_\rho (\epsilon^a \eta^\rho_{Aa} L^A),$$

which is equivalent to the generalized Trautman strong conservation law

$$\partial_\mu \theta^\mu_{(S)}[\epsilon^a] = \partial_\mu [\theta^\mu_{(W)}[\epsilon^a] + \epsilon^a \eta^\mu_{Aa} L^A] \equiv 0,$$

$$\Downarrow$$

$$\theta^\mu_{(S)}[\epsilon^a] = \partial_\nu\, U^{[\mu\nu]}[\epsilon^a],$$
$$\theta^\mu_{(W)}[\epsilon^a] = \epsilon^a t^\mu_a + \partial_\rho \epsilon^a t_a^{\mu\rho} + \partial_\nu \partial_\rho \epsilon^a \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_A} \eta^\rho_{Aa} + \partial_\nu V^{[\mu\nu]} =$$
$$= \theta^\mu_{(S)}[\epsilon^a] - \epsilon^a \eta^\mu_{Aa} L^A = \partial_\nu U^{[\mu\nu]}[\epsilon^a] - \epsilon^a \eta^\mu_{Aa} L^A, \qquad \text{(D6)}$$

where we introduced the superpotential $U^{[\mu\nu]}[\epsilon^a] = -U^{[\nu\mu]}[\epsilon^a]$ and the arbitrariness $V^{[\mu\nu]}$ implicit in the first line of Eq.(D3).

Let us now assume that we have a subgroup $G_p$ of the general group $G_{\infty q}$ so that one can write $\epsilon^a(x) = \epsilon^{\bar a}\zeta^a_{\bar a}(x)$, with $\epsilon^{\bar a}$, $\bar a$=1,..,p, the constant parameters of $G_p$. Then one has the following restrictions to $G_p$:

$\delta x^\mu = \epsilon^{\bar a}\zeta^a_{\bar a}(x)\xi^\mu_a(x) = \epsilon^{\bar a}\hat\xi^\mu_{\bar a}(x)$, $\delta_o\phi_A = \epsilon^{\bar a}(\zeta^a_{\bar a}(x)\eta_{Aa}(x) + \partial_\nu\zeta^a_{\bar a}(x)\eta^\nu_{Aa}(x)) = \epsilon^{\bar a}\hat\eta_{A\bar a}(x)$, $F^\mu = \epsilon^{\bar a}(\zeta^a_{\bar a}F^\mu_a + \partial_\nu\zeta^a_{\bar a}F^{\mu\nu}_a) = \epsilon^{\bar a}\hat F^\mu_{\bar a}$.

Then, one gets [162] the following improper weak conservation laws

$$\partial_\mu t^\mu_{\bar a} \stackrel{def}{=} \partial_\mu[(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A})\hat\eta_{A\bar a} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_A}\partial_\nu\hat\eta_{A\bar a} + \mathcal{L}\hat\xi^\mu_{\bar a} - \hat F^\mu_{\bar a}] \equiv -\hat\eta_{A\bar a}L^A \circeq 0, \quad \text{(D7)}$$

and, by using the contracted Bianchi identities, also the strong ones

$$\partial_\mu(t^\mu_{\bar a} - \zeta^a_{\bar a}\eta^\mu_{A\bar a}L^A) \equiv 0.$$

Let us now apply the second Noether theorem to the Lagrangian densities $\mathcal{L}_H = \sqrt{{}^4g}\,{}^4R$ with ${}^4g_{\mu\nu}$ as independent variables and with equations of motion $L^{\mu\nu} = \sqrt{{}^4g}\,{}^4G^{\mu\nu} \circeq 0$. One has strict invariance $\delta_o\mathcal{L}_H \equiv 0$, i.e. $F^\mu_H \equiv 0$, under the infinitesimal diffeomorphisms [see Eqs.(31) of I]

$$\delta x^\mu(x) = \epsilon^\mu(x),$$

$$\begin{aligned}
&\delta_o\,{}^4g_{\mu\nu}(x) = -\epsilon^\rho\partial_\rho\,{}^4g_{\mu\nu}(x) - \partial_\sigma\epsilon^\rho(x)[\delta^\sigma_\nu\,{}^4g_{\mu\rho}(x) + \delta^\sigma_\mu\,{}^4g_{\rho\nu}(x)] = \\
&+\delta\,{}^4g_{\mu\nu}(x) - \epsilon^\rho(x)\partial_\rho\,{}^4g_{\mu\nu}(x) = \mathcal{L}_{-\epsilon^\alpha\partial_\alpha}\,{}^4g_{\mu\nu}(x). \qquad \text{(D8)}
\end{aligned}$$

From Eqs.(3) of I and from $\partial\,{}^4\Gamma^\lambda_{\rho\sigma}/\partial\partial_\mu\,{}^4g_{\alpha\beta} = \frac{1}{2}\,{}^4g^{\lambda\tau}(\delta^\mu_\rho\delta^{\alpha\beta}_{(\tau\sigma)} + \delta^\mu_\sigma\delta^{\alpha\beta}_{(\tau\rho)} - \delta^\mu_\tau\delta^{\alpha\beta}_{(\rho\sigma)})$ with $\delta^{\alpha\beta}_{(\mu\nu)} = \frac{1}{2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu)$, one gets

$$\begin{aligned}
&\frac{\partial\mathcal{L}_H}{\partial\partial_\mu\partial_\nu\,{}^4g_{\alpha\beta}} = \sqrt{{}^4g}[\frac{1}{2}({}^4g^{\mu\alpha}\,{}^4g^{\nu\beta} + {}^4g^{\mu\beta}\,{}^4g^{\nu\alpha}) - {}^4g^{\mu\nu}\,{}^4g^{\alpha\beta}],\\
&\frac{\partial\mathcal{L}_H}{\partial\partial_\mu\,{}^4g_{\alpha\beta}} = \sqrt{{}^4g}[{}^4g^{\mu\rho}({}^4g^{\alpha\sigma}\,{}^4\Gamma^\beta_{\rho\sigma} + {}^4g^{\beta\sigma}\,{}^4\Gamma^\alpha_{\rho\sigma}) - {}^4g^{\alpha\rho}\,{}^4g^{\beta\sigma}\,{}^4\Gamma^\mu_{\rho\sigma}] +\\
&+{}^4g^{\mu\alpha}\partial_\rho(\sqrt{{}^4g}\,{}^4g^{\rho\beta}) + {}^4g^{\mu\beta}\partial_\rho(\sqrt{{}^4g}\,{}^4g^{\rho\alpha}) - {}^4g^{\alpha\beta}\partial_\rho(\sqrt{{}^4g}\,{}^4g^{\rho\mu}),\\
&\frac{\partial\mathcal{L}_H}{\partial\,{}^4g_{\alpha\beta}} = \sqrt{{}^4g}[-{}^4G^{\alpha\beta} + \partial_\rho\,{}^4g_{\eta\tau}\,{}^4g^{\mu\nu}\,{}^4\Gamma^\sigma_{\mu\nu}({}^4g^{\rho\eta}\,\delta^{(\alpha}_\sigma\,{}^4g^{\beta)\tau} - \delta^\tau_\sigma\,{}^4g^{\rho(\alpha}\,{}^4g^{\beta)\eta}) -\\
&-{}^4g^{\rho(\alpha}\,{}^4g^{\beta)\delta}\,{}^4g^{\mu\nu}\partial_\rho({}^4g_{\delta\sigma}\,{}^4\Gamma^\sigma_{\mu\nu}) - \frac{1}{4}\partial_\rho\,{}^4g_{\gamma\delta}\,{}^4g^{\mu\nu}\,{}^4\Gamma^\eta_{\mu\nu}({}^4g^{\gamma\delta}\,{}^4g^{\rho(\alpha}\,\delta^{\beta)}_\eta + \delta^\rho_\eta\,{}^4g^{\gamma(\alpha}\,{}^4g^{\beta)\delta}) +\\
&+\frac{1}{2}\,{}^4g^{\mu\nu}({}^4g^{\alpha\gamma}\,{}^4g^{\beta\delta}\partial_\mu\partial_\nu\,{}^4g_{\gamma\delta} - 2\,{}^4g^{\eta(\alpha}\,{}^4g^{\beta)\gamma}\,{}^4g^{\tau\delta}\partial_\mu\,{}^4g_{\eta\tau}\partial_\nu\,{}^4g_{\gamma\delta}) +\\
&+{}^4g^{\mu\nu}\,{}^4\Gamma^\eta_{\mu\sigma}\,{}^4\Gamma^\tau_{\nu\rho}({}^4g^{\rho(\alpha}\,\delta^{\beta)}_\eta\delta^\sigma_\tau + \delta^\rho_\eta\,{}^4g^{\sigma(\alpha}\,\delta^{\beta)}_\tau)]. \qquad \text{(D9)}
\end{aligned}$$

The weak and strong conservation laws are respectively

$$\partial_\mu\, {}_H\theta^\mu_{(W)}[\epsilon^\rho] \equiv [\epsilon^\rho \partial_\rho\, {}^4g_{\mu\nu} + \partial_\sigma \epsilon^\rho (\delta^\sigma_\nu\, {}^4g_{\mu\rho} + \delta^\sigma_\mu\, {}^4g_{\rho\nu})] \sqrt{{}^4g}\, {}^4G^{\mu\nu} \stackrel{\circ}{=} 0,$$

$$\partial_\mu\, {}_H\theta^\mu_{(S)\rho}[\epsilon^\rho] = \partial_\mu [{}_H\theta^\mu_{(W)}[\epsilon^\rho] - \epsilon^\rho (\delta^\mu_\alpha\, {}^4g_{\rho\beta} + \delta^\mu_\beta\, {}^4g_{\rho\alpha}) \sqrt{{}^4g}\, {}^4G^{\alpha\beta}] \equiv 0,$$

$$\begin{aligned}
{}_H\theta^\mu_{(S)}[\epsilon^\rho] &= \partial_\nu U^{[\mu\nu]}[\epsilon^\rho],\\
{}_H\theta^\mu_{(W)}[\epsilon^\rho] &= \epsilon^\rho t^\mu_\rho + \partial_\sigma \epsilon^\rho t^{\mu\sigma}_\rho - \partial_\nu \partial_\sigma \epsilon^\rho \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}} (\delta^\sigma_\alpha\, {}^4g_{\beta\rho} + \delta^\sigma_\beta\, {}^4g_{\alpha\rho}) + \partial_\nu V^{[\mu\nu]} =\\
&= -[\epsilon^\rho \partial_\rho\, {}^4g_{\alpha\beta} + \partial_\sigma \epsilon^\rho (\delta^\sigma_\alpha\, {}^4g_{\beta\rho} + \delta^\sigma_\beta\, {}^4g_{\alpha\rho})] (\frac{\partial \mathcal{L}_H}{\partial \partial_\mu\, {}^4g_{\alpha\beta}} - \partial_\nu \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}}) -\\
&- \partial_\nu [\epsilon^\rho \partial_\rho\, {}^4g_{\alpha\beta} + \partial_\sigma \epsilon^\rho (\delta^\sigma_\alpha\, {}^4g_{\beta\rho} + \delta^\sigma_\beta\, {}^4g_{\alpha\rho})] \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}} + \epsilon^\mu \sqrt{{}^4g}\, {}^4R =\\
&= \partial_\nu U^{[\mu\nu]} + \epsilon^\rho (\delta^\mu_\alpha\, {}^4g_{\rho\beta} + \delta^\mu_\beta\, {}^4g_{\rho\alpha}) \sqrt{{}^4g}\, {}^4G^{\alpha\beta}.
\end{aligned} \tag{D10}$$

Here $t^\mu_\rho$ is the candidate for the energy-momentum pseudo-tensor once one has done a choice for $U^{[\mu\nu]}$ and $V^{[\mu\nu]}$.

The Noether identities (D3) are satisfied with

$$\begin{aligned}
{}_Ht^{\mu\nu}_\rho &= -(\delta^\nu_\alpha\, {}^4g_{\beta\rho} + \delta^\nu_\beta\, {}^4g_{\alpha\rho}) (\frac{\partial \mathcal{L}_H}{\partial \partial_\mu\, {}^4g_{\alpha\beta}} - \partial_\nu \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}}) -\\
&- [\delta^\nu_\sigma \partial_\rho\, {}^4g_{\alpha\beta} + \partial_\sigma (\delta^\nu_\alpha\, {}^4g_{\beta\rho} + \delta^\nu_\beta\, {}^4g_{\alpha\rho})] \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}},\\
{}_Ht^\mu_\rho &= -\partial_\rho\, {}^4g_{\alpha\beta} (\frac{\partial \mathcal{L}_H}{\partial \partial_\mu\, {}^4g_{\alpha\beta}} - \partial_\nu \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}}) -\\
&- \partial_\nu \partial_\rho\, {}^4g_{\alpha\beta} \frac{\partial \mathcal{L}_H}{\partial \partial_\mu \partial_\nu\, {}^4g_{\alpha\beta}} + \delta^\mu_\rho \sqrt{{}^4g}\, {}^4R,\\
\partial_\mu\, {}_Ht^{\mu\nu}_\rho &+ {}_Ht^\nu_\rho \equiv (\delta^\mu_\alpha\, {}^4g_{\rho\beta} + \delta^\mu_\beta\, {}^4g_{\rho\alpha}) \sqrt{{}^4g}\, {}^4G^{\alpha\beta} \stackrel{\circ}{=} 0,\\
\partial_\mu\, {}_Ht^\mu_\rho &\equiv \partial_\rho\, {}^4g_{\alpha\beta} \sqrt{{}^4g}\, {}^4G^{\alpha\beta} \stackrel{\circ}{=} 0,
\end{aligned} \tag{D11}$$

and the contracted Bianchi identities are

$${}^4\nabla_\alpha\, {}^4G^{\alpha\beta} = \frac{1}{\sqrt{{}^4g}} \partial_\alpha (\sqrt{{}^4g}\, {}^4G^{\alpha\beta}) + {}^4\Gamma^\beta_{\alpha\gamma}\, {}^4G^{\alpha\gamma} \equiv 0. \tag{D12}$$

The conclusion of this discussion based on the second Noether theorem for the generally covariant Hilbert action is that any vector field

$$\xi^\mu = \sqrt{{}^4g}\, \epsilon^\mu$$

(it is more convenient to use $\xi^\mu$ rather than $\epsilon^\mu$ as a parameter) generates a one-parameter group of diffeomorphisms, which gives rise

to a "weak" $[\partial_\mu\, \theta^\mu_{(W)}[\xi] \stackrel{\circ}{=} 0]$ and a "strong" $[\partial_\mu\, \theta^\mu_{(S)}[\xi] \equiv 0]$ conservation law.

Any strongly conserved quantity $\theta^{\mu}_{(S)}[\xi]$ can be written in the form $\theta^{\mu}_{(S)}[\xi] = \partial_\nu U^{[\mu\nu]}[\xi]$ [whatever the transformation properties of $\xi^\mu$ are; $\xi^\mu$ may also not be a 4-vector], where $U^{[\mu\nu]} = -U^{[\nu\mu]}$ is a "superpotential" defined up to another antisymmetric quantity $V^{[\mu\nu]} = -V^{[\nu\mu]}$. One can show that

$$\theta^{\mu}_{(S)}[\xi] = \partial_\nu U^{[\mu\nu]}[\xi] \equiv \theta^{\mu}_{(W)}[\xi] + {}^4G^{\mu}{}_{\nu}\xi^\nu \overset{\circ}{=} \theta^{\mu}_{(W)}[\xi] + {}^4T^{\mu}{}_{\nu}\xi^\nu$$

so that by changing the superpotential, $U^{[\mu\nu]} \mapsto U'^{[\mu\nu]} = U^{[\mu\nu]} + V^{[\mu\nu]}$, one has $\theta^{\mu}_{(W)} \mapsto \theta'^{\mu}_{(W)} = \theta^{\mu}_{(W)} + \partial_\nu V^{[\mu\nu]}$. Two superpotentials are especially important

A) "Komar [164] covariant superpotential" ($\theta^{\mu}_{(S)}[\xi]$ is a vector density):

$$_{(K)}U^{[\mu\nu]}[\xi] = \frac{c^3}{8\pi G}\sqrt{{}^4g}({}^4\nabla^\mu\, \xi^\nu - {}^4\nabla^\nu\, \xi^\mu). \tag{D13}$$

In this case the associated weak conservation law $\partial_{\mu}\, {}_{(K)}\theta^{\mu}_{(W)} \overset{\circ}{=} 0$ is the divergence of a tensor density $[\partial_{\mu}\, {}_{(K)}\theta^{\mu}_{(W)} = \sqrt{{}^4g}\, {}^4\nabla_{\mu}\, {}_{(K)}\hat{\theta}^{\mu}_{(W)} \overset{\circ}{=} 0]$. Therefore, this is a coordinate-free expression which depends, however, on the choice of the vector field $\xi^\mu$, which, in asymptotically flat spacetimes, is chosen to tend to asymptotic Killing vectors for the evaluation of the global conserved quantities. Let us remark that ${}_{(K)}\theta^{\mu}_{(W)}$ contains the second derivatives of the metric.

B) "Bergmann [173] superpotential"

$$\begin{aligned}
&{}_{(B)}U^{[\mu\nu]}[\xi] = {}_{(F)}U_\lambda{}^{[\mu\nu]}\xi^\lambda,\\
&{}_{(F)}U_\lambda{}^{[\mu\nu]} = -\frac{c^3}{16\pi G}\frac{1}{\sqrt{{}^4g}}\, {}^4g_{\lambda\rho}\partial_\gamma[{}^4g({}^4g^{\mu\gamma}\, {}^4g^{\nu\rho} - {}^4g^{\nu\gamma}\, {}^4g^{\mu\rho})], \quad or\\
&{}_{(F)}U^{\lambda[\mu\nu]} = {}^4g^{\lambda\rho}\, {}_{(F)}U_\rho{}^{[\mu\nu]} = \frac{c^3}{16\pi G}\frac{1}{\sqrt{{}^4g}}\partial_\beta({}^4\hat{g}^{\nu\beta}\, {}^4\hat{g}^{\mu\lambda} - {}^4\hat{g}^{\nu\lambda}\, {}^4\hat{g}^{\mu\beta}) =\\
&= \frac{c^3}{16\pi G}\frac{1}{\sqrt{{}^4g}}\partial_\beta \mathcal{T}^{\nu\lambda\beta\mu},
\end{aligned} \tag{D14}$$

where ${}_{(F)}U_\lambda{}^{[\mu\nu]}$ is the Freud superpotential [157] (the notation of Eqs.(D19)-(D21) is used).

All the known gravitational pseudotensors or complexes can be derived from the superpotentials ${}_{(K)}U^{[\mu\nu]}[\xi]$ and ${}_{(B)}U^{[\mu\nu]}[\xi]$.

1) "Einstein canonical pseudo-tensor" - If in ${}_{(B)}U^{[\mu\nu]}[\xi]$ one chooses $\xi^\mu$ to be an object with constant components in any coordinate system, one gets [see also Eq.(D20)]

$$\begin{aligned}
&\theta^{\mu}_{(S)}[\xi] = \xi^\lambda\partial_\nu\, {}_{(F)}U_\lambda{}^{[\mu\nu]} = \xi^\lambda\, {}_{(E)}\theta_\lambda{}^\mu \overset{\circ}{=} \theta^{\mu}_{(W)}[\xi] + {}^4T^{\mu}{}_{\lambda}\xi^\lambda = \xi^\lambda({}_{(E)}t_\lambda{}^\mu + {}^4T_\lambda{}^\mu),\\
&{}_{(E)}\theta_\nu{}^\mu \overset{\circ}{=} {}_{(E)}t_\nu{}^\mu + {}^4T_\nu{}^\mu,\\
&\partial_\mu\, {}_{(E)}\theta_\nu{}^\mu \equiv 0, \qquad {}_{(E)}\theta^{\nu\mu} = {}^4g^{\nu\alpha}\, {}_{(E)}\theta_\alpha{}^\mu \not= {}_{(E)}\theta^{\mu\nu},\\
&{}_{(E)}t_\nu{}^\mu = -\delta^\mu_\nu\, {}^4Z + \partial_\nu\, {}^4g_{\alpha\beta}\frac{\partial\, {}^4Z}{\partial\partial_\mu\, {}^4g_{\alpha\beta}}, \qquad {}^4Z = {}^4R - \partial_\rho(\partial_\mu\, {}^4g_{\alpha\beta}\frac{\partial\, {}^4R}{\partial\partial_\rho\partial_\mu\, {}^4g_{\alpha\beta}}),\\
&{}_{(E)}t_\nu{}^\mu = 2\sqrt{{}^4g}\, {}^4G_\nu{}^\mu + \frac{c^3}{16\pi G}\partial_\alpha(\frac{{}^4g_{\mu\sigma}}{\sqrt{{}^4g}}\mathcal{T}^{\alpha\nu\beta\sigma}),
\end{aligned} \tag{D15}$$

where ${}_{(E)}t_\nu{}^\mu$ is Einstein's energy-momentum pseudo-tensor [174] and ${}_{(E)}\theta_\nu{}^\mu$ is a tensor density of weight 1.

2) "Landau-Lifschitz [62] symmetric pseudo-tensor" - It is obtained from ${}_{(B)}U^{[\mu\nu]}[\xi]$ by choosing the "covariant" components $\xi_\mu/\sqrt{{}^4g}$ to be constant [see also Eq.(D19)]:

$$
\begin{aligned}
&{}_{(L)}\theta^{\mu\nu} \equiv \partial_\gamma(\sqrt{{}^4g}\,{}^4g^{\mu\delta}\,{}_{(F)}U_\delta{}^{[\nu\gamma]}) \,\mathrel{\overset{\circ}{=}}\, {}_{(L)}t^{\mu\nu} + \sqrt{{}^4g}\,{}^4T^{\mu\nu},\\
&{}_{(L)}\theta^{\mu\nu} = {}_{(L)}\theta^{\nu\mu}, \qquad \partial_\nu\,{}_{(L)}\theta^{\mu\nu} \equiv 0,\\
&{}_{(L)}t^{\mu\nu} = 2\,{}^4g\,{}^4G^{\mu\nu} + \frac{c^3}{16\pi G}\partial_\rho\partial_\sigma \mathcal{T}^{\alpha\nu\beta\mu}.
\end{aligned} \tag{D16}
$$

${}_{(L)}\theta^{\mu\nu}$ is a tensor density of weight 2 and ${}_{(L)}t^{\mu\nu}$ is the Landau-Lifschitz pseudo-tensor [see Refs. [175] for some of its applications], which contains no second derivatives of the metric and gives meaningful results only in an asymptotically flat Cartesian coordinate system. It was found trying to rewrite ${}^4\nabla_\mu\,{}^4T^{\mu\nu} \mathrel{\overset{\circ}{=}} 0$ in the form $\partial_\mu[{}_{(L)}t^{\mu\nu} + \sqrt{{}^4g}\,{}^4T^{\mu\nu}] \mathrel{\overset{\circ}{=}} 0$. Starting from Einstein's equations ${}^4T^{\mu\nu} \mathrel{\overset{\circ}{=}} \frac{c^3}{8\pi G}({}^4R^{\mu\nu} - \frac{1}{2}{}^4g^{\mu\nu}\,{}^4R)$, one rewrites them as

$$
{}_{(L)}t^{\mu\nu} + \sqrt{{}^4g}\,{}^4T^{\mu\nu} \mathrel{\overset{\circ}{=}} \partial_\rho h^{\mu[\nu\rho]} \text{ with } h^{\mu[\nu\rho]} = -h^{\mu[\rho\nu]} = \tfrac{c^3}{16\pi G}\partial_\sigma[\sqrt{{}^4g}({}^4g^{\mu\nu}\,{}^4g^{\rho\sigma} - {}^4g^{\mu\rho}\,{}^4g^{\nu\sigma})],
$$

which imply $\partial_\nu\partial_\rho h^{\mu[\nu\rho]} = 0$. Then one gets

$$
\begin{aligned}
{}_{(L)}t^{\mu\nu} &= \frac{c^3}{16\pi G}\sqrt{{}^4g}[(2\,{}^4\Gamma^\rho_{\tau\sigma}\,{}^4\Gamma^\gamma_{\rho\gamma} - {}^4\Gamma^\rho_{\tau\gamma}\,{}^4\Gamma^\gamma_{\sigma\rho} - {}^4\Gamma^\rho_{\tau\rho}\,{}^4\Gamma^\gamma_{\sigma\gamma})({}^4g^{\mu\tau}\,{}^4g^{\nu\sigma} - {}^4g^{\mu\nu}\,{}^4g^{\tau\sigma}) +\\
&+{}^4g^{\mu\tau}\,{}^4g^{\sigma\rho}({}^4\Gamma^\nu_{\tau\gamma}\,{}^4\Gamma^\gamma_{\sigma\rho} + {}^4\Gamma^\nu_{\sigma\rho}\,{}^4\Gamma^\gamma_{\tau\gamma} - {}^4\Gamma^\nu_{\rho\gamma}\,{}^4\Gamma^\gamma_{\tau\sigma} - {}^4\Gamma^\nu_{\tau\sigma}\,{}^4\Gamma^\gamma_{\rho\gamma}) +\\
&+{}^4g^{\nu\tau}\,{}^4g^{\sigma\rho}({}^4\Gamma^\mu_{\tau\gamma}\,{}^4\Gamma^\gamma_{\sigma\rho} + {}^4\Gamma^\mu_{\sigma\rho}\,{}^4\Gamma^\gamma_{\tau\gamma} - {}^4\Gamma^\mu_{\rho\sigma}\,{}^4\Gamma^\gamma_{\tau\sigma} - {}^4\Gamma^\mu_{\tau\sigma}\,{}^4\Gamma^\gamma_{\rho\gamma}) +\\
&+{}^4g^{\tau\sigma}\,{}^4g^{\rho\gamma}({}^4\Gamma^\mu_{\tau\rho}\,{}^4\Gamma^\nu_{\sigma\gamma} - {}^4\Gamma^\mu_{\tau\sigma}\,{}^4\Gamma^\nu_{\rho\gamma})] =\\
&= \frac{c^3}{16\pi G}[\partial_\rho\,{}^4\hat{g}^{\mu\nu}\,\partial_\gamma\,{}^4\hat{g}^{\gamma\rho} - \partial_\rho\,{}^4\hat{g}^{\mu\rho}\partial_\gamma\,{}^4\hat{g}^{\nu\gamma} + \frac{1}{2}{}^4g^{\mu\nu}\,{}^4g_{\rho\sigma}\,\partial_\sigma\,{}^4\hat{g}^{\rho\delta}\,\partial_\delta\,{}^4\hat{g}^{\sigma\gamma} -\\
&-({}^4g^{\mu\rho}\,{}^4g_{\gamma\delta}\partial_\sigma\,{}^4\hat{g}^{\nu\delta}\,\partial_\rho\,{}^4\hat{g}^{\gamma\sigma} + {}^4g^{\nu\rho}\,{}^4g_{\gamma\delta}\partial_\sigma\,{}^4\hat{g}^{\mu\delta}\,\partial_\rho\,{}^4\hat{g}^{\gamma\sigma}) + {}^4g_{\rho\sigma}\,{}^4g^{\gamma\delta}\partial_\gamma\,{}^4\hat{g}^{\mu\rho}\partial_\delta\,{}^4\hat{g}^{\nu\sigma} +\\
&+\frac{1}{8}(2\,{}^4g^{\mu\rho}\,{}^4g^{\nu\sigma} - {}^4g^{\mu\nu}\,{}^4g^{\rho\sigma})(2\,{}^4g_{\gamma\delta}\,{}^4g_{\alpha\beta} - {}^4g_{\delta\alpha}\,{}^4g_{\gamma\beta})\partial_\rho\,{}^4\hat{g}^{\gamma\beta}\partial_\sigma\,{}^4\hat{g}^{\delta\alpha}].
\end{aligned} \tag{D17}
$$

The Einstein pseudo-tensor and Landau-Lifschitz complex are the only complexes that are homogeneous quadratic in first derivatives of the metric tensor. Therefore, both complexes can be made to vanish along any one geodesic by a suitable choice of coordinates. For a spacetime that is asymptotically flat either in spacelike or null directions, the Landau-Lifschitz and Einstein superpotentials give exactly the same definition of energy-momentum. Both require an asymptotically rectangular coordinate system to be meaningful. Once defined, both transform as free vectors under the asymptotic symmetry group.

3) "Møller's energy-momentum complex" [165] - It is obtained by ${}_{(K)}U^{[\mu\nu]}[\xi]$ by putting $\xi^\mu = const.$ [Komar found his superpotential trying to generalize Møller's one]:

$$
{}_{(M)}\theta_\mu{}^\nu = \partial_\gamma\,{}_{(M)}U_\mu{}^{[\nu\gamma]}, \qquad {}_{(M)}U_\mu{}^{[\nu\gamma]} = \frac{c^3}{8\pi G}\sqrt{{}^4g}\,{}^4g^{\nu\alpha}\,{}^4g^{\gamma\beta}(\partial_\alpha\,{}^4g_{\mu\beta} - \partial_\beta\,{}^4g_{\mu\alpha}). \tag{D18}
$$

While all the previous pseudotensors depend on the metric and its first derivatives, Møller complex also depends on the second derivatives of the metric.

4) Lorentz [160] chose $\theta^{\mu}_{(S)}[\xi] = 0$ and identified $\frac{c^3}{8\pi G}{}^4G^{\mu\nu} \circeq 0$ with the energy momentum of the gravitational field in absence of matter [149]. This also was the point of view of Levi-Civita [176].

Let us remark that in Ref. [159] there is the following derivation of three pseudotensors based only on the use of the Einstein equations. Given Einstein's equations of motion $\frac{\partial \mathcal{L}_E}{\partial\, {}^4g_{\mu\nu}} - \partial_\alpha \frac{\partial \mathcal{L}_E}{\partial \partial_\alpha\, {}^4g_{\mu\nu}} \circeq -\, \frac{8\pi G}{c^3}\, {}^4T^{\mu\nu}$ and defined the tensor density of weight +2,

$$\mathcal{T}^{\alpha\nu\beta\mu} = {}^4\hat{g}^{\alpha\beta}\, {}^4\hat{g}^{\mu\nu} - {}^4\hat{g}^{\alpha\mu}\, {}^4\hat{g}^{\beta\nu} = -\mathcal{T}^{\nu\alpha\beta\mu} = -\mathcal{T}^{\alpha\nu\mu\beta} = \mathcal{T}^{\beta\mu\alpha\nu}$$

$[\mathcal{T}^{\alpha\nu\beta\mu} + \mathcal{T}^{\alpha\beta\mu\nu} + \mathcal{T}^{\alpha\mu\nu\beta} = 0]$, one has the following three ways of rewriting the equations of motion:

1) the Landau-Lifschitz [62] weak conservation law

$$\begin{aligned}
&-\frac{1}{2\sqrt{{}^4g}}\partial_\alpha\partial_\beta \mathcal{T}^{\alpha\nu\beta\mu} + (terms\, homogeneous\, quadratic\, in\, \partial\, {}^4g) \circeq -\, \frac{8\pi G}{c^3}\, {}^4T^{\mu\nu},\\
&\frac{c^3}{16\pi G}\partial_\alpha\partial_\beta \mathcal{T}^{\alpha\nu\beta\mu} \circeq {}_{(L)}t^{\mu\nu} + \sqrt{{}^4g}\, {}^4T^{\mu\nu}, \quad with \quad {}_{(L)}t^{\mu\nu} = {}_{(L)}t^{\nu\mu},\\
&\partial_\nu[{}_{(L)}t^{\mu\nu} + \sqrt{{}^4g}\, {}^4T^{\mu\nu}] \circeq 0;
\end{aligned} \tag{D19}$$

2) the Einstein weak conservation law: by taking the factor ${}^4g_{\mu\sigma}/\sqrt{{}^4g}$ through $\partial_\alpha$ one gets

$$\begin{aligned}
&-\frac{1}{2}\partial_\alpha[\frac{{}^4g_{\mu\sigma}}{\sqrt{{}^4g}}\partial_\beta \mathcal{T}^{\alpha\nu\beta\sigma}] + (terms\, homogeneous\, quadratic\, in\, \partial\, {}^4g) \circeq -\, \frac{8\pi G}{c^3}\, {}^4T^{\nu}_{\mu},\\
&\frac{c^3}{16\pi G}\partial_\alpha[\frac{{}^4g_{\mu\sigma}}{\sqrt{{}^4g}}\partial_\beta \mathcal{T}^{\alpha\nu\beta\sigma}] \circeq {}_{(E)}t^{\nu}_{\mu} + {}^4T^{\nu}_{\mu},\\
&\partial_\nu[{}_{(E)}t^{\nu}_{\mu} + {}^4T^{\nu}_{\mu}] \circeq 0;
\end{aligned} \tag{D20}$$

3) the Bergmann-Thompson [161] weak conservation law: by taking the factor $1/\sqrt{{}^4g}$ through $\partial_\alpha$ one gets

$$\begin{aligned}
&-\frac{1}{2}\partial_\alpha[\frac{1}{\sqrt{{}^4g}}\partial_\beta \mathcal{T}^{\alpha\nu\beta\mu}] + (terms\, homogeneous\, quadratic\, in\, \partial\, {}^4g) \circeq -\, \frac{8\pi G}{c^3}{}^4T^{\mu\nu},\\
&\frac{c^3}{16\pi G}\partial_\alpha[\frac{1}{\sqrt{{}^4g}}\partial_\beta \mathcal{T}^{\alpha\nu\beta\mu}] \circeq {}_{(BT)}t^{\mu\nu} + {}^4T^{\mu\nu}, \qquad {}_{(BT)}t^{\mu\nu} \neq {}_{(BT)}t^{\nu\mu},\\
&\partial_\nu({}_{(BT)}t^{\mu\nu} + {}^4T^{\mu\nu}) \circeq 0, \qquad \partial_\mu({}_{(BT)}t^{\mu\nu} + {}^4T^{\mu\nu}) \neq 0.
\end{aligned} \tag{D21}$$

A whole class of weak conservation equations may be obtained by taking the factor $(\sqrt{{}^4g})^m$, for any integer m, with or without a factor ${}^4g_{\mu\sigma}$ through the $\partial_\alpha$. Some of these conservation equations are among those considered by Goldberg [163]. All the conserved quantities in the class contain the source term ${}^4T^{\mu\nu}$ added to a homogeneous quadratic function of the first derivatives of the metric; in a sense all the conservation equations in the class are equivalent to one another.

The total energy contained in a finite or infinite 3-volume V with 2-dimensional boundary $\partial V$ is [$d^3\Sigma_\mu$ and $d^2\Sigma_{\mu\nu}$ denote the volume and area elements of V and $\partial V$ respectively]

$$
\begin{aligned}
\psi[\xi] &= \int_V d^3\Sigma_\mu\, \theta^\mu_{(S)}[\xi] = \int_{\partial V} d^2\Sigma_{\mu\nu}\, U^{[\mu\nu]}[\xi] \equiv \\
&\equiv \int_V d^3\Sigma_\mu\, [\theta^\mu_{(W)}[\xi] + {}^4G^\mu{}_\nu\, \xi^\nu] \stackrel{\circ}{=} \int_V d^3\Sigma_\mu\, [\theta^\mu_{(W)}[\xi] + {}^4T^\mu{}_\nu \xi^\nu].
\end{aligned} \tag{D22}
$$

Here $\psi[\xi]$ is a scalar if $\theta^\mu_{(S)}[\xi]$ is a vector density as in Komar's case. However, this scalar does not characterize intrinsically the region V unless we find a way of picking out a "privileged" $\xi^\mu$. Since the action $S_H$ is invariant under all the one-parameter groups generated by a vector filed $\xi^\mu$, one gets an infinity of weak conservation laws [173], as well as of strong ones.

If the spacetime is asymptotically flat, one can take for $\xi^\mu$ a vector field which coincides with a Killing vector at spatial infinity (if the flat spacetime is Minkowski one, one has 10 asymptotic Killing vectors satisfying $\xi_{\mu;\nu} + \xi_{\nu;\mu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$) and one can show that $\psi[\xi] = \int_{\partial V \rightarrow \infty} d^2\Sigma_{\mu\nu}\, U^{[\mu\nu]}[\xi]$ does not depend on the choice of $\xi^\mu$ at finite distances, so that 10 global quantities can be constructed if $\xi^\mu(\vec{x}, x^o) \rightarrow_{|\vec{x}| \rightarrow \infty} a^\mu + \omega^\mu{}_\nu x^\nu$ $[\omega_{\mu\nu} = -\omega_{\nu\mu}]$. Since $\xi^\mu$ generates an one-parameter group of diffeomorphisms, here are hidden the restrictions that one has to impose on the global structure of the diffeomorphism group.

For $\xi^\mu(x) \rightarrow a^\mu$ one gets $\psi[\xi] = a^\mu P_\mu$ with $P_\mu = \int_{\partial V} d^2\Sigma_{\alpha\beta\, (B)} U^{[\alpha\beta]}{}_\mu \stackrel{\circ}{=} \int_V d^3\Sigma_\alpha\, [\theta^\alpha_{(W)\mu} + {}^4T^\alpha{}_\mu]$ [if $\theta^\alpha_{(W)} = \theta^\alpha_{(W)\mu}\xi^\mu$] in the case of the Bergmann superpotential. This formula holds in particular for the Einstein canonical superpotential: ${}_{(E)}P_\mu[V] = \int_{\partial V} d^2\Sigma_{\alpha\beta\, (F)} U^{[\alpha\beta]}{}_\mu \stackrel{\circ}{=} \int_V d^3\Sigma_{\alpha\, (E)}\theta_\mu{}^\alpha$, and the Einstein-Klein theorem says that the surface integral is convergent in asymptotically flat coordinates ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + O(r^{-1})$, $\partial_\gamma\, {}^4g_{\mu\nu} = O(r^{-2})$ [isolated nonradiating system], is independent from the spacelike hypersurface $\Sigma$ containing the volume V [the conservation law gives $\int_{V'} d^3\Sigma_{\alpha\, (E)}\theta_\mu{}^\alpha - \int_V d^3\Sigma_{\alpha\, (E)}\theta_\mu{}^\alpha = \int_{S_\infty} d^3\Sigma_{\alpha\, (E)} t_\mu{}^\alpha = 0$, because the matter is confined and, with the assumed boundary conditions on the 4-metric, the Einstein pseudo-tensor ${}_{(E)}t_\nu{}^\mu$ is of order $O(r^{-4})$ and does not contribute with a gravitational field 4-momentum; $S_\infty$ is a timelike 3-surface at space infinity connecting $\Sigma'$ and $\Sigma$ limits of $V'$ and $V$ respectively], is unaltered by coordinate transformations with the same asymptotic limit and that ${}_{(E)}P_\mu$ transforms as a covariant vector under linear transformations. By using Cartesian coordinates at infinity, one gets $P_o = m$ and $P_i = 0$ for the Schwarzschild solution.

The previous asymptotic boundary conditions on the 4-metric are satisfied by the static fields produced by matter confined to a finite volume, but in general they exclude the possibility of radiation. Comparison with electrodynamics suggests that radiation fields in general relativity should be characterized by $\partial_\gamma\, {}^4g_{\mu\nu} = O(r^{-1})$ rather that by $\partial_\gamma\, {}^4g_{\mu\nu} = O(r^{-2})$. However, if the integral of ${}_{(E)}t_\nu{}^\mu$ over $S_\infty$ does not vanish, the argument used to prove the Einstein-Klein theorem is no longer valid and the meaning of $P_\mu[\Sigma]$ becomes obscure. If the radiation goes on with a finite rate from $x^o = -\infty$ , one cannot even expect the integrals $P_\mu[\Sigma]$ to be convergent [the space is filled with an infinite amount of energy in the form of radiation]. Nevertheless, if the system remains quiescent till, say, $x^o = 0$, then radiates for a while and again quiets down, one can give a reasonable prescription for calculating the total energy and its rate of change with the same procedure used in the linearized theory [169]: one calculates the energy in coordinates systems which asymptotically satisfy the harmonic condition $\partial_\beta(\sqrt{{}^4g}\, {}^4g^{\alpha\beta}) = 0$ [177]. Let us assume that the gravitational filed in question defines a scalar field u(x) whose gradient, $k_\mu = \partial_\mu u$, is null and diverging. This allows us to introduce a "luminosity distance" r through ${}^4\nabla_\mu(r^{-2}\, k^\mu) = 0$ [see Bondi and Sachs [178,179]]. Now we can formulate the boundary conditions which constitute a generalization

of Sommerfeld's radiation conditions to gravitational fields: there exist coordinate systems and functions $i_{\mu\nu} = O(r^{-1})$ such that ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + O(r^{-1})$, $\partial_\gamma\, {}^4g_{\mu\nu} = i_{\mu\nu} k_\gamma + O(r^{-2})$, $(i_{\mu\nu} - \frac{1}{2}{}^4\eta_{\mu\nu}\, {}^4\eta^{\gamma\delta} i_{\gamma\delta}) = O(r^{-2})$. The expression $P_\mu(u) = lim_{u=const.,r\rightarrow\infty} \int_{\partial V} d^2\Sigma_{\alpha\beta\, (F)} U_\mu{}^{\alpha\beta}$, if they converge, give the total energy and momentum of the system as function of the retarded time u; $P_o$ is called the Bondi mass [see Refs. [180], where after a reformulation of the Landau-Lifschitz complex in a manifest covariant way with a background metric following Refs. [181,159], its covariant formulation at null infinity in asymptotically flat spacetimes is given and the Bondi 4-momentum is recovered]. Our boundary conditions ensure that the 1/r terms in the integrand cancel out and so make plausible the existence of $P_\mu(u)$. Moreover, the $P_\mu(u)$ are invariant under coordinate transformations which preserve the boundary conditions and reduce to the identity for $r \rightarrow \infty$. If radiation goes on only for a finite interval of time, $P_\mu[\Sigma]$ is well defined and equal to $P_\mu(u = -\infty)$. The pseudo-tensor in the wave zone is given by the same expression as the canonical tensor in the linear theory [169]. Under the further assumption $\partial_\gamma\partial_\delta\, {}^4g_{\mu\nu} = O(r^{-1})$, the previous boundary conditions allow to prove that the 1/r part of the curvature tensor is of Petrov's type II null.

Formally, the Komar superpotential differs from the Einstein pseudo-tensor: it gives the same results at spatial infinity, but at null infinity it must be modified in order to give the Bondi mass [182,183] in the case of the Kerr metric [184]. As noted in Ref. [185], the conserved Komar quantities $K = \int_\Sigma d^3\Sigma_\mu\, \partial_\nu\, {}_{(K)}U^{[\mu\nu]}[\xi] = \frac{c^3}{8\pi G} \int_{\partial\Sigma} d^2\Sigma_{\mu\nu} \sqrt{{}^4g} [{}^4\nabla_\mu \xi^\nu - {}^4\nabla_\nu \xi^\mu]$ [$\Sigma$ is a spacelike hypersurface] have the following properties: i) K is equal to the mass M for Schwarzschild and Kerr black holes if $\xi^\mu = (1,0,0,0)$ in coordinates $(x^o, r, \theta, \varphi)$; ii) for Kerr black holes and $\xi^\mu = (0,0,0,1)$ in Boyer-Linquist coordinates, K is "twice" the angular momentum J [this is the "anomalous factor 2"]; iii) for the radiating asymptotic solution of Ref. [186], if $\xi^\mu$ is asymptotically (1,0,0,0) in radiative coordinates $(u = x^o - r, r, \theta, \varphi)$, K is not Bondi's mass $M(u,\theta)$ but rather $M + \frac{1}{2}c\dot{c}$ in Bondi's notation and the correction to Komar's integral to get rid of the $c\dot{c}$ term has been found in Ref. [182].

In Ref. [159] by using a Minkowski background metric it is shown that, if Trautman's boundary conditions are satisfied, the same 4-momentum $P_\mu[\Sigma]$ is obtained starting from the Einstein, the Landau-Lifschits and Bergmann-Thomson [161] pseudotensors [a system is defined as non-radiative if the 4-momentum is the same for all spacelike hypersurfaces $\Sigma$]. Then, in the second of Refs. [159], it is analyzed the case of the solution corresponding to an isolated axi-symmetric system generating gravitational waves found by Bondi, van der Burg and Metzner [186]. These authors were able to show that the behaviour of the system was fully determined by a single function, the "news" function, and initial conditions specified on the light cone. They used a definition of mass of the system such that in the static case the definition led to the correct quantity; their mass is the Bondi mass. In the approach with the background metric one recovers the same value of the mass [ due to symmetry, in this case there are only the energy and the momentum along the symmetry axis]. One uses expansions such that an outgoing radiation condition of Sommerfeld type probably equivalent to the Trautmann conditions] is automatically satisfied.

All this discussion of the conservation laws in the generally covariant approach based on the Hilbert action is not directly connected with the weak and strong ADM charges, as can be seen in the more recent review given in Ref. [13] of the definitions of energy-momentum at spatial and null infinity in asymptotically flat spacetimes. Essentially one has that the variation in time of the Bondi energy at null infinity may be interpreted as defining the flux

of energy carried away to infinity by gravitational radiation and that this agrees with the energy flux computed from the Landau-Lifschitz pseudotensor with an appropriately chosen background and, then, with the energy flux of the linearized theory.

Instead, in the case of a timelike Killing field $\xi^\mu$ asymptotically orthogonal to the Cauchy surface the strong ADM energy-moemtum 4-vector gives the same definition for the total mass of all stationary asymptotically flat at spatial infinity spacetimes as one would get from the Komar superpotential. More work will be needed to get a consistent picture englobing all these properties for tetrad gravity. In particular, the results of this paper show that one has to study in detail the effect of gravitational radiation in the tetrad gravity reformulation of the Christodoulou and Klainermann spacetimes and to find a bridge to the Bondi results at null infinity. Since the natural formalism for discussing null infinity is the Newman-Penrose one [131], in Appendix F we give the definition of a set of null tetrads natural from the Hamiltonian point of view to be used as a starting point to find the Hamiltonian version of the Newman-Penrose formalism.

## APPENDIX E: 4-TENSORS IN VOID SPACETIMES.

From Eqs.(103) and Appendices D and E of II, we get the following expression for various 3- and 4-tensors in void spacetimes in the 3-orthogonal gauges $[N(\tau,\vec{\sigma}) \approx -\epsilon + n(\tau,\vec{\sigma}) \approx -\epsilon$, $N_r(\tau,\vec{\sigma}) \approx \hat{n}_r(\tau,\vec{\sigma}|q,\rho] \approx 0$ for $\rho(\tau,\vec{\sigma}) \approx 0$; see Section VI for a complete gauge fixing; we use either $q$, $\rho$ or $\phi = e^{q/2}$, $\pi_\phi = 2\phi^{-1}\rho$ with $q = 0$ ($\phi = 1$) and $\rho = \pi_\phi = 0$ as the realization of the flat limit; when $\rho = 0$ we have $n = 0$]

$$\begin{aligned}
{}^3\hat{\tilde{\pi}}^r_{(a)}(\tau,\vec{\sigma}) &= \frac{1}{3}\int d^3\sigma_1 \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1;\tau|q,0]\rho(\tau,\vec{\sigma}_1) \ \rightarrow_{\rho\rightarrow 0}\ 0,\\
{}^3\hat{K}_{rs} &= \frac{\epsilon}{4k}\sum_u (\delta_{rs}\delta_{(a)s} + \delta_{su}\delta_{(a)r} - \delta_{rs}\delta_{(a)u})\, {}^3\hat{\tilde{\pi}}^u_{(a)} \ \rightarrow_{\rho\rightarrow 0}\ 0,\\
{}^3\hat{\omega}_{r(a)} &= \epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u}\partial_u q \ \rightarrow_{q\rightarrow 0}\ 0,\\
{}^3\hat{\Omega}_{rs(a)} &= \epsilon_{(a)(b)(c)}\sum_u \delta_{(c)u}[\delta_{(b)s}\partial_u\partial_r q - \delta_{(b)r}\partial_u\partial_s q] +\\
&+ \frac{1}{2}[\delta_{(a)(b)}\epsilon_{(c)(d)(e)} - \delta_{(a)(c)}\epsilon_{(b)(d)(e)} + \delta_{(a)(d)}\epsilon_{(e)(c)(b)} - \delta_{(a)(e)}\epsilon_{(d)(c)(b)}]\partial_u q\partial_v q \rightarrow_{q\rightarrow 0} 0,\\
{}^3\hat{R}_{rusv} &= (\delta_{rv}\delta_{su} - \delta_{rs}\delta_{uv})e^{4q}\sum_n (\partial_n q)^2 +\\
&+ e^{2q}\{\delta_{rv}[\partial_s\partial_u q - \partial_s q\partial_u q] - \delta_{rs}[\partial_v\partial_u q - \partial_v q\partial_u q] +\\
&+ \delta_{su}[\partial_v\partial_r q - \partial_v q\partial_r q] - \delta_{uv}[\partial_s\partial_r q - \partial_s q\partial_r q]\}\rightarrow_{q\rightarrow 0} 0,\\
{}^3\hat{R}_{uv} &= -\partial_u\partial_v q + \partial_u q\partial_v q - \delta_{uv}e^{2q}\sum_n [2e^{2q}(\partial_n q)^2 + \partial_n^2 q - (\partial_n q)^2] \rightarrow_{q\rightarrow 0} 0,\\
{}^3\hat{R} &= -6\sum_u (\partial_u q)^2 - 4e^{-2q}\sum_u [\partial_u^2 q - (\partial_u q)^2] \rightarrow_{q\rightarrow 0} 0,
\end{aligned} \tag{E1}$$

$$\begin{aligned}
{}^4\hat{\Gamma}^\tau_{\tau\tau} &\,\circeq\, \frac{1}{N}\partial_\tau N \ \rightarrow_{n\rightarrow 0,\tilde{\lambda}_\tau\rightarrow\epsilon}\ 0,\\
{}^4\hat{\Gamma}^\tau_{r\tau} &= {}^4\hat{\Gamma}^\tau_{\tau r} = \frac{1}{N}\partial_r N \ \rightarrow_{n\rightarrow 0}\ 0,\\
{}^4\hat{\Gamma}^\tau_{rs} &= {}^4\hat{\Gamma}^\tau_{sr} = -\frac{\epsilon}{4kN}\sum_u {}^3G_{o(a)(b)(c)(d)}\delta_{(a)r}\delta_{(b)s}\delta_{(c)u}\, {}^3\hat{\tilde{\pi}}^u_{(d)} \ \rightarrow_{\rho\rightarrow 0}\ 0,\\
{}^4\hat{\Gamma}^u_{\tau\tau} &\,\circeq\, \phi^{-4}N\partial_u N \ \rightarrow_{n\rightarrow 0}\ 0,\\
{}^4\hat{\Gamma}^u_{r\tau} &= {}^4\hat{\Gamma}^u_{\tau r} = -\frac{\epsilon N}{4k}\sum_{uv}\phi^{-4}\,{}^3G_{o(a)(b)(c)(d)}\delta^u_{(a)}\delta_{(b)r}\delta_{(c)v}\, {}^3\hat{\tilde{\pi}}^v_{(d)} \ \rightarrow_{\rho\rightarrow 0}\ 0,\\
{}^4\hat{\Gamma}^u_{rs} &= {}^3\hat{\Gamma}^u_{rs} = -\delta_{uv}\sum_v \delta^u_v\partial_v q + \delta^u_r\partial_s q + \delta^u_s\partial_r q \ \rightarrow_{q\rightarrow 0}\ 0,
\end{aligned}$$

$$\begin{aligned}
{}^4\hat{\overset{\circ}{\omega}}_{\tau(o)(a)} &= -{}^4\hat{\overset{\circ}{\omega}}_{\tau(a)(o)} = -\epsilon\phi^{-2}\sum_r \delta_{(a)r}\partial_r N \ \rightarrow_{n\rightarrow 0}\ 0,\\
{}^4\hat{\overset{\circ}{\omega}}_{\tau(a)(b)} &= -{}^4\hat{\overset{\circ}{\omega}}_{\tau(b)(a)} \circeq 0,\\
{}^4\hat{\overset{\circ}{\omega}}_{r(o)(a)} &= -{}^4\hat{\overset{\circ}{\omega}}_{r(a)(o)} = -\frac{1}{4k}\phi^{-2}\sum_u {}^3G_{o(a)(b)(c)(d)}\delta_{(b)r}\delta_{(c)u}\, {}^3\hat{\tilde{\pi}}^u_{(d)} \ \rightarrow_{\rho\rightarrow 0}\ 0,\\
{}^4\hat{\overset{\circ}{\omega}}_{r(a)(b)} &= -{}^4\hat{\overset{\circ}{\omega}}_{r(b)(a)} = -\epsilon\, {}^3\hat{\omega}_{r(a)(b)} \ \rightarrow_{q\rightarrow 0}\ 0,
\end{aligned}$$

$$
\begin{aligned}
{}^4\hat{\overset{\circ}{\Omega}}_{rs(a)(b)} &= -\epsilon\,[{}^3\hat{\Omega}_{rs(a)(b)} + \frac{\phi^{-4}}{4k}\,{}^3G_{o(a)(c)(d)(e)}\,{}^3G_{o(b)(f)(g)(h)}\cdot\\
&\quad \sum_{uv}(\delta_{(c)r}\delta_{(f)s} - \delta_{(c)s}\delta_{(f)r})\delta_{(d)u}\,{}^3\hat{\tilde{\pi}}^u_{(e)}\,\delta_{(g)v}\,{}^3\hat{\tilde{\pi}}^v_{(h)}] \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{\overset{\circ}{\Omega}}_{rs(o)(a)} &= \frac{1}{N}\phi^{-4}\sum_v \delta_{(a)v}({}^4\hat{R}_{\tau vrs} - N_{(b)}\sum_u \delta_{(b)u}\,{}^4\hat{R}_{uvrs}) =\\
&= \frac{1}{4k}\phi^{-2}\sum_u \delta_{(a)u}[({}^3G_{o(b)(c)(d)(e)}\,\delta_{(b)r}\delta_{(c)u}\delta_{(d)v}\,{}^3\hat{\tilde{\pi}}^v_{(e)})_{|s} -\\
&- ({}^3G_{o(b)(c)(d)(e)}\,\delta_{(b)s}\delta_{(c)u}\delta_{(d)v}\,{}^3\hat{\tilde{\pi}}^v_{(e)})_{|r}] \rightarrow_{\rho\rightarrow 0}\ 0,\\
{}^4\hat{\overset{\circ}{\Omega}}_{\tau r(a)(b)} &= \phi^{-4}\sum_{uv}\delta_{(a)u}\delta_{(b)v)}\,{}^4\hat{R}_{uv\tau r} \overset{\circ}{=}\\
&\overset{\circ}{=} -\epsilon\{\partial_\tau\,{}^3\hat{\omega}_{r(a)(b)} + \frac{1}{2}(\epsilon_{(a)(b)(c)}\epsilon_{(d)(e)(f)} - \epsilon_{(a)(b)(d)}\epsilon_{(c)(e)(f)})\cdot\\
&\quad \sum_s \phi^{-2}\delta_{(c)s}[\frac{\epsilon N}{4k}\phi^{-2}\sum_v\,{}^3G_{o(d)(l)(m)(n)}\,\delta_{(l)s}\delta_{(m)v}\,{}^3\hat{\tilde{\pi}}^v_{(n)} +\\
&+ N_{(l)}\phi^2\sum_u \delta_{(l)u}\partial_u(\delta_{(d)s}\phi^2) + \phi^2\sum_u\delta_{(d)u}\partial_s(N_{(l)}\delta_{(l)u}\phi^{-2}) +\\
&+ \phi^2\epsilon_{(d)(m)(n)}\hat{\mu}_{(m)}\delta_{(n)s} - N_{(g)}\phi^{-2}\sum_u\delta_{(g)u}\partial_u(\delta_{(d)s}\phi^2) -\\
&- \phi^2\sum_u\delta_{(d)u}\partial_s(N_{(g)}\delta_{(g)u}\phi^{-2})]^3\hat{\omega}_{r(e)(f)} + N_{(c)}\phi^{-2}\delta_{(c)s}\,[{}^3\hat{\omega}_s, {}^3\hat{\omega}_r]_{(a)(b)} +\\
&+ \frac{\epsilon}{4k}\phi^{-4}\sum_u\,{}^3G_{o(c)(d)(e)(f)}\delta_{(c)r}\delta_{(e)u}\,{}^3\hat{\tilde{\pi}}^u_{(f)}\\
&\quad (\delta_{(a)(d)}\delta_{(b)u} - \delta_{(b)(d)}\delta_{(a)u})\partial_u N +\\
&+ \frac{1}{(4k)^2}(\delta_{(a)(l)}\delta_{(b)(d)} - \delta_{(a)(d)}\delta_{(b)(l)})^3G_{o(d)(e)(f)(g)}\,{}^3G_{o(h)(l)(m)(n)}\cdot\\
&\cdot\ \phi^{-6}\sum_{wv}\delta_{(h)r}N_{(e)}\delta_{(f)w}\,{}^3\hat{\tilde{\pi}}^w_{(g)}\delta_{(m)v}\,{}^3\hat{\tilde{\pi}}^v_{(n)}\} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{\overset{\circ}{\Omega}}_{\tau r(o)(a)} &\overset{\circ}{=} \frac{1}{N}\,\phi^{-2}\sum_u\delta_{(a)u}[{}^4\hat{R}_{\tau u\tau r} - N_{(b)}\phi^{-2}\sum_s\delta_{(b)s}\,{}^4\hat{R}_{su\tau r}] \overset{\circ}{=}\\
&\overset{\circ}{=} -\epsilon\,\phi^{-2}\sum_s\delta_{(a)s}[\partial_\tau\,{}^3\hat{K}_{rs} + N_{|s|r} -\\
&- \frac{\epsilon}{4k}\sum_{uw}\,{}^3G_{o(c)(d)(e)(f)}\delta_{(d)u}\delta_{(e)w}\,{}^3\hat{\tilde{\pi}}^w_{(f)}\\
&\quad \{\delta_{(c)r}(N_{(b)}\phi^{-2}\delta_{(b)u})_{|s} + \delta_{(c)s}(N_{(b)}\phi^{-2}\delta_{(b)u})_{|r}\} -\\
&- \frac{\epsilon}{4k}\phi^2\sum_{usw}N_{(b)}\delta_{(b)u}\\
&\quad ({}^3G_{o(c)(d)(e)(f)}\,\delta_{(c)s}\delta_{(d)u}\delta_{(e)w}\,{}^3\hat{\tilde{\pi}}^w_{(f)})_{|r}\,] \rightarrow_{q,\rho,n\rightarrow 0}\ 0,
\end{aligned}
$$

$$
{}^4\hat{R}_{rsuv} = \phi^4\delta_{(a)r}\delta_{(b)s}\,{}^4\hat{\overset{\circ}{\Omega}}_{uv(a)(b)} =
$$

$$
\begin{aligned}
&= -{}^3\hat{R}_{rsuv} + \frac{N^2}{16k^2} \sum_{tw} {}^3G_{o(a)(b)(c)(d)}\, {}^3G_{o(e)(f)(g)(h)}\\
&\cdot \delta_{(a)r}\delta_{(e)s}(\delta_{(b)u}\delta_{(f)v} - \delta_{(b)v}\delta_{(f)u})\delta_{(c)t}\delta_{(g)w}\, {}^3\hat{\tilde{\pi}}^t_{(d)}\, {}^3\hat{\tilde{\pi}}^w_{(h)} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{R}_{\tau r u v} &= N\phi^2 \delta_{(a)r}\, {}^4\hat{\overset{\circ}{\Omega}}_{uv(o)(a)} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{R}_{\tau r \tau s} &= N\phi^2 \delta_{(a)r}\, {}^4\hat{\overset{\circ}{\Omega}}_{\tau s(o)(a)} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{R}_{\tau\tau} &= -\epsilon\phi^{-4} \sum_r {}^4\hat{R}_{r\tau r\tau} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{R}_{\tau r} &= {}^4\hat{R}_{r\tau} = -\epsilon\phi^{-4} \sum_u {}^4\hat{R}_{u\tau u r} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{R}_{rs} &= {}^4\hat{R}_{sr} = \frac{\epsilon}{N^2} {}^4\hat{R}_{\tau r\tau s} - \epsilon\phi^{-4} \sum_u {}^4\hat{R}_{urus} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{R} &= \frac{\epsilon}{N^2} {}^4\hat{R}_{\tau\tau} - \epsilon\phi^{-4} \sum_r {}^4\hat{R}_{rr} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{C}_{rsuv} &= {}^4\hat{R}_{rsuv} + \frac{\epsilon}{2}[\phi^4(\delta_{rv}\, {}^4\hat{R}_{su} - \delta_{ru}\, {}^4\hat{R}_{sv}) +\\
&+ \phi^4(\delta_{su}\, {}^4\hat{R}_{rv} - \delta_{sv}\, {}^4\hat{R}_{ru})] +\\
&+ \frac{1}{6}\phi^8(\delta_{ru}\delta_{sv} - \delta_{rv}\delta_{su}) {}^4\hat{R} \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{C}_{\tau ruv} &= {}^4\hat{R}_{\tau ruv} + \frac{\epsilon}{2}\phi^4(\delta_{ru}\, {}^4\hat{R}_{\tau v} - \delta_{rv}\, {}^4\hat{R}_{\tau u}) \rightarrow_{q,\rho\rightarrow 0}\ 0,\\
{}^4\hat{C}_{\tau r\tau s} &= {}^4\hat{R}_{\tau r\tau s} + \frac{1}{2}(N^2\, {}^4\hat{R}_{rs} - \epsilon\phi^4\delta_{rs}\, {}^4\hat{R}_{\tau\tau}) -\\
&- \frac{1}{6}N^2\phi^4\delta_{rs}\, {}^4\hat{R} \rightarrow_{q,\rho\rightarrow 0}\ 0.
\end{aligned}
\tag{E2}
$$

Using Eqs.(69) and (70) of I, Ashtekar's variables become [we give the limits for $\phi = e^{q/2} = 1$ and $\rho = 0$]

$$
\begin{aligned}
&{}^3\tilde{h}^r_{(a)}(\tau,\vec{\sigma}) \mapsto {}^3\hat{\tilde{h}}^r_{(a)}(\tau,\vec{\sigma}) = \delta^r_{(a)}\phi^4(\tau,\vec{\sigma}) \rightarrow \delta^r_{(a)},\\
&{}^3A_{(a)r}(\tau,\vec{\sigma}) \mapsto {}^3\hat{A}_{(a)r}(\tau,\vec{\sigma}) = \frac{1}{6k}\phi^4(\tau,\vec{\sigma})\\
&\int d^3\sigma_1\, \mathcal{K}^r_{(a)s}(\vec{\sigma},\vec{\sigma}_1,\tau|\phi,0]\, [\phi^{-2}\rho](\tau,\vec{\sigma}_1) +\\
&+ 2i\epsilon_{(a)(b)(c)}\delta_{(b)r}\delta_{(c)u}\partial_u ln\, \phi(\tau,\vec{\sigma}) \rightarrow \frac{1}{6k}.
\end{aligned}
\tag{E3}
$$

## APPENDIX F: $\Sigma_\tau$-ADAPTED NULL TETRADS.

The 3+1 splitting of $M^4$ with spacelike slices $\Sigma_\tau$ not only identifies a timelike vector field, namely the unit normal to $\Sigma_\tau$ [see Eqs.(40) of I; $l^A = \epsilon\, {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(o)} = \frac{\epsilon}{N}(1; -{}^3e^r_{(a)} N_{(a)})$, ${}^4g_{AB} l^A l^B = \epsilon$, $l_A = {}^4_{(\Sigma)}\check{\tilde{E}}^{(o)}_A = (N; \vec{0})$; by construction it is surface forming: $\frac{1}{N} l_A d\sigma^A$ is a closed differential 1-form], but also a spacelike vector field $\mathcal{N}^A$ tangent to $\Sigma_\tau$ by means of the shift functions:

$$\mathcal{N}^A = \frac{N_{(a)}}{\sqrt{\sum_{(c)} N^2_{(c)}}}\, {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(a)} = \frac{1}{\sqrt{\sum_{(c)} N^2_{(c)}}} (0; {}^3e^r_{(a)} N_{(a)}) \qquad , {}^4g_{AB}\mathcal{N}^A\mathcal{N}^B = -\epsilon,$$
$$\mathcal{N}_A = -\epsilon N_{(a)}\, {}^4_{(\Sigma)}\check{\tilde{E}}^{(a)}_A = -\epsilon\Big(\sqrt{\textstyle\sum_{(c)} N^2_{(c)}}; {}^3e_{(a)r} \frac{N_{(a)}}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big).$$

The two directions $l^A$, $\mathcal{N}^A$ are intrinsically selected by the gauge nature of the lapse and shift functions in every 3+1 splitting.

Let us remark that the vector field $\mathcal{N}^A(\tau,\vec{\sigma})$ is not in general surface forming, namely the associated differential 1-form $\mathcal{N}(\tau,\vec{\sigma}) = \mathcal{N}_A(\tau,\vec{\sigma}) d\sigma^A$ is not proportional to a closed 1-form. Since we have

$$d\Big[-\frac{\epsilon\mathcal{N}}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big] = \partial_r\Big[\frac{{}^3e_{(a)s} N_{(a)}}{\sum_{(c)} N^2_{(c)}}\Big] d\sigma^r \wedge d\sigma^s,$$

its vanishing implies

$$\partial_\tau \frac{N_r}{\vec{N}^2} = 0 \qquad , \partial_r \frac{N_s}{\vec{N}^2} = \partial_s \frac{N_r}{\vec{N}^2},$$

with $N_r = {}^3e_{(a)r} N_{(a)}$, $\vec{N}^2 = \sum_{(c)} N^2_{(c)} = {}^3g^{rs} N_r N_s$. Therefore, the condition for having $\mathcal{N}^A(\tau,\vec{\sigma})$ surface-forming (zero vorticity) is the choice of a coordinate system on $\Sigma_\tau$ such that Eq.(36) [for $N_r = n_r$, $N_{(as)r} = 0$] implies

$$N_r = \vec{N}^2 \partial_r f \text{ with } \partial_\tau f = 0.$$

3-orthogonal coordinates do not imply this property of $\mathcal{N}^A$. In the coordinate systems for $\Sigma_\tau$ in which $\mathcal{N}^A$ is surface forming there is a second foliation of $M^4$ with timelike hypersurfaces $\Xi_\zeta$ (if $\zeta$ is the parameter labelling the leaves: $\mathcal{N} = k d\zeta$), and the intersection $\Sigma_\tau \cap \Xi_\zeta = S_{\tau\zeta}$ is a 2-surface whose tangent space in each point is a 2-plane spanned by the two spacelike vectors perpendicular to $l^A$ and $\mathcal{N}^A$ in that point (see later on the vector field $M^A$, $\bar{M}^A$). Therefore, in these special coordinate systems for $\Sigma_\tau$ one could perform a 2+2 decomposition of $M^4$ along the lines of Refs. [141]. The study of the Shanmugadhasan canonical transformation and of the superhamiltonian constraint in these coordinates should allow to identify the analog of the natural gauge fixing $\rho \approx 0$ in the 3-orthogonal gauges.

Therefore, in each point of $\Sigma_\tau$ we can select two orthogonal vectors $\overset{\circ}{V}{}^{(\alpha)}$ and $S^{(\alpha)}$ in the tangent plane:

one timelike $\overset{\circ}{V}{}^{(\alpha)} = l^A(\tau,\vec{\sigma})\, {}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_A(\tau,\vec{\sigma}) = (1; \vec{0}) \qquad , {}^4\eta_{(\alpha)(\beta)} \overset{\circ}{V}{}^{(\alpha)} \overset{\circ}{V}{}^{(\beta)} = \epsilon,$

and the other spacelike $S^{(\alpha)}(\tau,\vec{\sigma}) = \mathcal{N}^A(\tau,\vec{\sigma})\, {}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_A(\tau,\vec{\sigma}) = (0; \frac{N_{(a)}}{\sqrt{\sum_{(c)} N^2_{(c)}}})(\tau,\vec{\sigma})$, ${}^4\eta_{(\alpha)(\beta)}[S^{(\alpha)}S^{(\beta)}](\tau,\vec{\sigma}) = -\epsilon$, ${}^4\eta_{(\alpha)(\beta)}\overset{\circ}{V}{}^{(\alpha)} S^{(\beta)}(\tau,\vec{\sigma}) = 0$.

Then in each point of $\Sigma_\tau$ we can define the following two null tangent vectors

$$
\begin{aligned}
\mathcal{K}^{(\alpha)}(\tau,\vec{\sigma}) &= \sqrt{\frac{1}{2}\sum_{(c)} N^2_{(c)}(\tau,\vec{\sigma})}\Big[\overset{\circ}{V}{}^{(\alpha)} + S^{(\alpha)}(\tau,\vec{\sigma})\Big] = \frac{1}{\sqrt{2}}(\sqrt{\sum_{(c)} N^2_{(c)}}; N_{(a)})(\tau,\vec{\sigma}) = \\
&= \sqrt{\frac{1}{2}\sum_{(c)} N^2_{(c)}(\tau,\vec{\sigma})}\Big(l^A + \mathcal{N}^A\Big)(\tau,\vec{\sigma})\, {}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_A(\tau,\vec{\sigma}),\\
\mathcal{L}^{(\alpha)}(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{2}\sum_{(c)} N^2_{(c)}(\tau,\vec{\sigma})}\Big[\overset{\circ}{V}{}^{(\alpha)} - S^{(\alpha)}(\tau,\vec{\sigma})\Big] = \\
&= \frac{1}{\sqrt{2}\sum_{(c)} N^2_{(c)}(\tau,\vec{\sigma})}(\sqrt{\sum_{(c)} N^2_{(c)}}; -N_{(a)})(\tau,\vec{\sigma}) = \\
&= \frac{1}{\sqrt{2}\sum_{(c)} N^2_{(c)}(\tau,\vec{\sigma})}\Big(l^A - \mathcal{N}^A\Big)(\tau,\vec{\sigma})\, {}^4_{(\Sigma)}\check{\tilde{E}}^{(\alpha)}_A(\tau,\vec{\sigma}),\\
&\quad {}^4\eta_{(\alpha)(\beta)}[\mathcal{K}^{(\alpha)}\mathcal{K}^{(\beta)}](\tau,\vec{\sigma}) = {}^4\eta_{(\alpha)(\beta)}[\mathcal{L}^{(\alpha)}\mathcal{L}^{(\beta)}](\tau,\vec{\sigma}) = 0,\\
&\quad {}^4\eta_{(\alpha)(\beta)}[\mathcal{K}^{(\alpha)}\mathcal{L}^{(\beta)}](\tau,\vec{\sigma}) = \epsilon. \qquad\qquad (F1)
\end{aligned}
$$

The null vector $\mathcal{K}^{(\alpha)}(\tau,\vec{\sigma})$ may be obtained from the reference vector $\overset{(o)}{\mathcal{K}}{}^{(\alpha)} = \omega(1;001)$ [$\omega$ is a constant with the dimensions of the shift functions] by means of the standard Wigner helicity boost [187] [$(\lambda) = (1), (2)$]

$$
{}_H L^{(\alpha)}{}_{(\beta)}(\mathcal{K}, \overset{(o)}{\mathcal{K}}) = \begin{pmatrix}
\frac{1}{2}(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}) & 0 & \frac{1}{2}(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}) \\
\frac{1}{2}(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}})\frac{N_{(\lambda)}}{\sqrt{\sum_{(c)} N^2_{(c)}}} & \delta^{(\lambda)}_{(\lambda')} - \frac{N_{(\lambda)}N_{(\lambda')}}{\sqrt{\sum_{(c)} N^2_{(c)}}(\sqrt{\sum_{(c)} N^2_{(c)}}+N_{(3)})} & \frac{1}{2}(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}})\frac{N_{(\lambda)}}{\sqrt{\sum_{(c)} N^2_{(c)}}} \\
\frac{1}{2}(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}})\frac{N_{(3)}}{\sqrt{\sum_{(c)} N^2_{(c)}}} & -\frac{N_{(\lambda')}}{\sqrt{\sum_{(c)} N^2_{(c)}}} & \frac{1}{2}(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}})\frac{N_{(3)}}{\sqrt{\sum_{(c)} N^2_{(c)}}}
\end{pmatrix},
$$

$$\mathcal{K}^{(\alpha)} = {}_HL^{(\alpha)}{}_{(\beta)}(\mathcal{K}, \overset{(o)}{\mathcal{K}}) \overset{(o)}{\mathcal{K}}{}^{(\beta)}. \tag{F2}$$

The columns of ${}_HL^{(\alpha)}{}_{(\beta)}(\mathcal{K}, \overset{(o)}{\mathcal{K}})$ define a flat helicity tetrad

$$
\begin{aligned}
&{}_H\epsilon^{(\alpha)}_{(\tilde{\alpha})}(N_{(c)}(\tau,\vec{\sigma})) = {}_HL^{(\alpha)}{}_{(\beta)=(\tilde{\alpha})}(\mathcal{K}, \overset{(o)}{\mathcal{K}})(\tau,\vec{\sigma}),\\
&{}^4\eta^{(\alpha)(\beta)} = {}_H\epsilon^{(\alpha)}_{(\tilde{\alpha})}\, {}^4\eta^{(\tilde{\alpha})(\tilde{\beta})}\, {}_H\epsilon^{(\beta)}_{(\tilde{\beta})},
\end{aligned}
$$

such that

$$
\begin{aligned}
\mathcal{K}^{(\alpha)}(\tau,\vec{\sigma}) &= \frac{\omega}{\sqrt{2}}\Big[{}_H\epsilon^{(\alpha)}_{(\tilde{o})} + {}_H\epsilon^{(\alpha)}_{(\tilde{3})}\Big](\tau,\vec{\sigma}),\\
\mathcal{L}^{(\alpha)}(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{2}\omega}\Big[{}_H\epsilon^{(\alpha)}_{(\tilde{o})} - {}_H\epsilon^{(\alpha)}_{(\tilde{3})}\Big](\tau,\vec{\sigma}),
\end{aligned}
$$

$${}^4\eta^{(\alpha)(\beta)} = 2\Big[\mathcal{K}^{(\alpha)}\mathcal{L}^{(\beta)} + \mathcal{K}^{(\beta)}\mathcal{L}^{(\alpha)}\Big](\tau,\vec{\sigma}) - \sum_{(\tilde{\lambda})=(\tilde{1})}^{(\tilde{2})} \Big[{}_H\epsilon^{(\alpha)}_{(\tilde{\lambda})}\, {}_H\epsilon^{(\beta)}_{(\tilde{\lambda})}\Big](\tau,\vec{\sigma}). \tag{F3}$$

With the transverse helicity polarization vectors ${}_H\epsilon^{(\alpha)}_{(\tilde{\lambda})}(\tau,\vec{\sigma})$, in each point we can build circular complex polarization vectors and then a null tetrad

$$
\begin{aligned}
\mathcal{M}^{(\alpha)}(\tau,\vec{\sigma}) &= {}_H\epsilon^{(\alpha)}_{(-)}(\tau,\vec{\sigma}) = \frac{1}{\sqrt{2}}\Big[{}_H\epsilon^{(\alpha)}_{(\tilde{1})} - i\, {}_H\epsilon^{(\alpha)}_{(\tilde{2})}\Big](\tau,\vec{\sigma}),\\
\bar{\mathcal{M}}^{(\alpha)}(\tau,\vec{\sigma}) &= {}_H\epsilon^{(\alpha)}_{(+)}(\tau,\vec{\sigma}) = \frac{1}{\sqrt{2}}\Big[{}_H\epsilon^{(\alpha)}_{(\tilde{1})} + i\, {}_H\epsilon^{(\alpha)}_{(\tilde{2})}\Big](\tau,\vec{\sigma}),
\end{aligned}
$$

$${}^4\eta^{(\alpha)(\beta)} = 2\Big[\mathcal{K}^{(\alpha)}\mathcal{L}^{(\beta)} + \mathcal{K}^{(\beta)}\mathcal{L}^{(\alpha)} - \Big(\mathcal{M}^{(\alpha)}\bar{\mathcal{M}}^{(\beta)} + \mathcal{M}^{(\beta)}\bar{\mathcal{M}}^{(\alpha)}\Big)\Big](\tau,\vec{\sigma}). \tag{F4}$$

See Ref. [187] for the covariance properties of the polarization vectors ${}_H\epsilon^{(\alpha)}_{(\tilde{\lambda})}(\tau,\vec{\sigma})$ under Lorentz transformations and Ref. [131] for the associated transformation properties of the null tetrad.

Now we can build a helicity $\Sigma_\tau$-adapted tetrad in $\Sigma_\tau$- adapted coordinates in each point of $M^4$

$$
\begin{aligned}
{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{\alpha})}(\tau,\vec{\sigma}) &= {}_H\epsilon^{(\alpha)}_{(\tilde{\alpha})}(\tau,\vec{\sigma})\; {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(\alpha)},\\
{}^4g^{AB}(\tau,\vec{\sigma}) &= {}^4\eta^{(\tilde{\alpha})(\tilde{\beta})}\Big[{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{\alpha})}\; {}^4_{H(\Sigma)}\check{\tilde{E}}^B_{(\tilde{\beta})}\Big](\tau,\vec{\sigma}),\\
{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{o})}(\tau,\vec{\sigma}) &= \frac{1}{2}\Big[\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big)\epsilon l^A +\\
&\quad + \Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big)\mathcal{N}^A\Big](\tau,\vec{\sigma})
\end{aligned}
$$

$$
\begin{aligned}
&\rightarrow_{\omega=\sqrt{\sum_{(c)} N^2_{(c)}}} \qquad \epsilon l^A(\tau,\vec{\sigma}),\\
{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{3})}(\tau,\vec{\sigma}) &= \frac{1}{2}\Big[\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big)\epsilon l^A +\\
&+\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big)\mathcal{N}^A\Big](\tau,\vec{\sigma})\\
&\rightarrow_{\omega=\sqrt{\sum_{(c)} N^2_{(c)}}} \qquad \mathcal{N}^A(\tau,\vec{\sigma}),\\
{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{\lambda})}(\tau,\vec{\sigma}) &= {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(a)=(\tilde{\lambda})}(\tau,\vec{\sigma}) - \frac{N_{(\tilde{\lambda})}[\mathcal{N}^A + {}^4_{(\Sigma)}\check{\tilde{E}}^A_{(3)}}{\sqrt{\sum_{(c)} N^2_{(c)}} + N_{(3)}}(\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
l^A(\tau,\vec{\sigma}) &= \frac{\epsilon}{2}\Big[\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big){}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{o})} +\\
&+\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big){}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{3})}\Big](\tau,\vec{\sigma})\\
&\rightarrow_{\omega=\sqrt{\sum_{(c)} N^2_{(c)}}} \qquad \epsilon\, {}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{o})}(\tau,\vec{\sigma}),\\
\mathcal{N}^A(\tau,\vec{\sigma}) &= \frac{1}{2}\Big[\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} - \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big){}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{o})} +\\
&+\Big(\frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\omega} + \frac{\omega}{\sqrt{\sum_{(c)} N^2_{(c)}}}\Big){}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{3})}\Big](\tau,\vec{\sigma})\\
&\rightarrow_{\omega=\sqrt{\sum_{(c)} N^2_{(c)}}} \qquad {}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{3})}(\tau,\vec{\sigma}),
\end{aligned}
\tag{F5}
$$

and then a natural intrinsic [i.e. dictated by canonical tetrad gravity itself] null tetrad (to be used for doing the transition to the Newman-Penrose formalism [131])

$$
\begin{aligned}
L^A(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{2}}\Big[{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{o})} + {}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{3})}\Big](\tau,\vec{\sigma}) = \frac{\sqrt{\sum_{(c)} N^2_{(c)}}}{\sqrt{2}\omega}(\epsilon l^A + \mathcal{N}^A)(\tau,\vec{\sigma}) =\\
&= \frac{1}{\sqrt{2}\omega N}\Big(\sqrt{\sum_{(c)} N^2_{(c)}}; (N - \sqrt{\sum_{(c)} N^2_{(c)}})^3 e^r_{(a)} N_{(a)}\Big)(\tau,\vec{\sigma})\\
&\rightarrow_{\omega=\sqrt{\sum_{(c)} N^2_{(c)}}} \qquad \frac{1}{\sqrt{2}}[\epsilon l^A + \mathcal{N}^A](\tau,\vec{\sigma}) =\\
&= \frac{1}{\sqrt{2}N\sqrt{\sum_{(c)} N^2_{(c)}}}\Big(\sqrt{\sum_{(c)} N^2_{(c)}}; (N - \sqrt{\sum_{(c)} N^2_{(c)}})^3 e^r_{(a)} N_{(a)}\Big)(\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
K^A(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{2}}\Big[{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{o})} - {}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{3})}\Big](\tau,\vec{\sigma}) = \frac{\omega}{\sqrt{2\sum_{(c)}N^2_{(c)}}}(\epsilon l^A - \mathcal{N}^A)(\tau,\vec{\sigma}) =\\
&= \frac{\omega}{\sqrt{2}N\sum_{(c)}N^2_{(c)}}\Big(\sqrt{\sum_{(c)}N^2_{(c)}}; -(N+\sqrt{\sum_{(c)}N^2_{(c)}})^3e^r_{(a)}N_{(a)}\Big)(\tau\vec{\sigma})\\
&\rightarrow_{\omega=\sqrt{\sum_{(c)}N^2_{(c)}}} \qquad \frac{1}{\sqrt{2}}[\epsilon l^A - \mathcal{N}^A](\tau,\vec{\sigma}) =\\
&= \frac{1}{\sqrt{2}N\sqrt{\sum_{(c)}N^2_{(c)}}}\Big(\sqrt{\sum_{(c)}N^2_{(c)}}; -(N+\sqrt{\sum_{(c)}N^2_{(c)}})^3e^r_{(a)}N_{(a)}\Big)(\tau\vec{\sigma}),\\
M^A(\tau,\vec{\sigma}) &= {}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(-)}(\tau,\vec{\sigma}) = \frac{1}{\sqrt{2}}\Big[{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{1})} - i\,{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{2})}\Big](\tau,\vec{\sigma}) =\\
&= (0; {}^3e^r_{(-)} - \frac{N_{(-)}}{\sqrt{\sum_{(c)}N^2_{(c)}}+N_{(3)}}[{}^3e^r_{(a)}\frac{N_{(a)}}{\sqrt{\sum_{(c)}N^2_{(c)}}} + {}^3e^r_{(3)}])(\tau,\vec{\sigma}),\\
\bar{M}^A(\tau,\vec{\sigma}) &= {}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(+)}(\tau,\vec{\sigma}) = \frac{1}{\sqrt{2}}\Big[{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{1})} + i\,{}^4_{H(\Sigma)}\check{\tilde{E}}^A_{(\tilde{2})}\Big](\tau,\vec{\sigma}) =\\
&= (0; {}^3e^r_{(+)} - \frac{N_{(+)}}{\sqrt{\sum_{(c)}N^2_{(c)}}+N_{(3)}}[{}^3e^r_{(a)}\frac{N_{(a)}}{\sqrt{\sum_{(c)}N^2_{(c)}}} + {}^3e^r_{(3)}])(\tau,\vec{\sigma}),
\end{aligned}
$$

$$
{}^4g^{AB}(\tau,\vec{\sigma}) = 2\Big[L^AK^B + L^BK^A - (M^A\bar{M}^B + M^B\bar{M}^A)\Big](\tau,\vec{\sigma}),
$$

$$
\begin{aligned}
L_A(\tau,\vec{\sigma}) &= \frac{\sqrt{\sum_{(c)}N^2_{(c)}}}{\sqrt{2}\omega}(\epsilon l_A + \mathcal{N}_A)(\tau,\vec{\sigma}) =\\
&= \frac{\epsilon\sqrt{\sum_{(c)}N^2_{(c)}}}{\sqrt{2}\omega}(N - \sqrt{\sum_{(c)}N^2_{(c)}}; -{}^3e_{(3)r})(\tau,\vec{\sigma}),\\
K_A(\tau,\vec{\sigma}) &= \frac{\omega}{\sqrt{2\sum_{(c)}N^2_{(c)}}}(\epsilon l_A - \mathcal{N}_A)(\tau,\vec{\sigma}) =\\
&= \frac{\epsilon\sqrt{\sum_{(c)}N^2_{(c)}}}{\sqrt{2}\omega}(N + \sqrt{\sum_{(c)}N^2_{(c)}}; +{}^3e_{(3)r})(\tau,\vec{\sigma}),\\
M_A(\tau,\vec{\sigma}) &= -\epsilon(0; {}^3e_{(-)r} - \frac{N_{(-)}}{\sqrt{\sum_{(c)}N^2_{(c)}}+N_{(3)}}[{}^3e_{(a)r}\frac{N_{(a)}}{\sqrt{\sum_{(c)}N^2_{(c)}}} + {}^3e_{(3)r}])(\tau,\vec{\sigma}),\\
\bar{M}_A(\tau,\vec{\sigma}) &= -\epsilon(0; {}^3e_{(+)r} - \frac{N_{(+)}}{\sqrt{\sum_{(c)}N^2_{(c)}}+N_{(3)}}[{}^3e_{(a)r}\frac{N_{(a)}}{\sqrt{\sum_{(c)}N^2_{(c)}}} + {}^3e_{(3)r}])(\tau,\vec{\sigma}),\\
{}^4g_{AB}(\tau,\vec{\sigma}) &= 2[L_AK_B + L_BK_A - (M_A\bar{M}_B + M_B\bar{M}_A)](\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= 2[\theta^{(L)}\otimes\theta^{(K)} + \theta^{(K)}\otimes\theta^{(L)} - (\theta^{(M)}\otimes\theta^{(\bar{M})} + \theta^{(\bar{M})}\otimes\theta^{(M)})],\\
&\theta^{(L)} = L_Ad\sigma^A, \quad \theta^{(K)} = K_Ad\sigma^A, \quad \theta^{(M)} = M_Ad\sigma^A, \quad \theta^{(\bar{M})} = \bar{M}_Ad\sigma^A, \qquad \text{(F6)}
\end{aligned}
$$

Then, for $\omega = \sqrt{\sum_{(c)} N^2_{(c)}}$ we get

$$
\begin{aligned}
{}^3g_{rs} &= -2[L_r K_s + L_s K_r - (M_r \bar{M}_s + M_s \bar{M}_s)] = \\
&= -2[l_r l_s - (\mathcal{N}_r \mathcal{N}_s + M_r \bar{M}_s + M_s \bar{M}_r)] = \\
&= 2[\mathcal{N}_r \mathcal{N}_s + M_r \bar{M}_s + M_s \bar{M}_r],
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= \epsilon[\theta^{(\tau)} \otimes \theta^{(\tau)} - 2(\theta^{(\mathcal{N})} \otimes \theta^{(\mathcal{N})} + \theta^{(M)} \otimes \theta^{(\bar{M})} + \theta^{(\bar{M})} \otimes \theta^{(M)})], \\
&\theta^{(\tau)} = N d\tau, \qquad \theta^{(\mathcal{N})} = \mathcal{N}_r (d\sigma^r + N^r d\tau), \qquad \theta^{(M)} = M_r (d\sigma^r + N^r d\tau). \qquad \text{(F7)}
\end{aligned}
$$

In the 3-orthogonal gauges we have

$$
\begin{aligned}
M_r = -\epsilon \phi^2 e^{\frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}}} \Big[ & \frac{1}{\sqrt{2}} (\delta_{(1)r} - i \delta_{(2)r}) - \\
& - \frac{N_{(-)}}{\sqrt{\sum_{(c)} N^2_{(c)}} + N_{(3)}} (\delta_{(a)r} \frac{N_{(a)}}{\sqrt{\sum_{(c)} N^2_{(c)}}} + \delta_{(3)r}) \Big]. \qquad \text{(F8)}
\end{aligned}
$$

Given the previous null tetrad one could find an associated spin basis (see Appendix C) $o^{\tilde{A}}$, $i^{\tilde{A}}$ for the Newman-Penrose formalism, so to visualize its 3+1 decomposition and its dependence on the Hamiltonian gauge variables.

Then, from the line element $-2(\theta^{(M)} \otimes \theta^{(\bar{M})} + \theta^{(\bar{M})} \otimes \theta^{(M)})$, modulo a 2-conformal factor, one should identify a 2-metric ${}^2g_{\bar{a}\bar{b}}$, which in suitable adapted coordinates should depend only on the two new canonical variables $r^{'}_{\bar{a}}$ of the gravitational field resulting from the Shanmugadhasan canonical transformation applied to these 3-coordinates for $\Sigma_\tau$ (the $r^{'}_{\bar{a}}$'s would be functions only of $\tau$ and of the 2-coordinates on $S_{\tau\zeta}$, since there would be no dynamics in $\zeta$). This would simultaneously implement the ideas of Ref. [140] and give an explicit realization of the statement of Christodoulou-Klainermann [7] on the independent degrees of freedom of the gravitational field.

Instead to make contact with the Newman-Penrose formalism for studying the asymptotic behaviour of the gravitational field at null infinity, one should look for a coordinate system on $\Sigma_\tau$ and for a gauge fixing to the superhamiltonian constraint such that the resulting lapse and shift functions $N = -\epsilon + n$, $N_r = n_r$ imply the existence of a foliation of $M^4$ with a one-parameter family of null hypersurfaces $\mathcal{Z}_u$ labelled by a parameter $u$ (a retarded time) such that $L_A d\sigma^A$ is proportional to $du$ (see Ref. [131]). For instance, by asking

$$du = d\tau - \frac{{}^3e_{(a)r} d\sigma^r}{N - \sqrt{\sum_{(c)} N^2_{(c)}}},$$

we get that $N$ and $N_{(a)}$ must satisfy

$$\partial_\tau \frac{{}^3e_{(a)r}}{N - \sqrt{\sum_{(c)} N^2_{(c)}}} \qquad , \partial_r \frac{{}^3e_{(a)s}}{N - \sqrt{\sum_{(c)} N^2_{(c)}}} = \partial_s \frac{{}^3e_{(a)r}}{N - \sqrt{\sum_{(c)} N^2_{(c)}}}.$$

Let us finally remark that the definitions of $\mathcal{N}^A(\tau, \vec{\sigma})$, of the null tetrad and of the previous construction become singular in synchronous 4-coordinates $[N_{(a)} = 0]$.

## APPENDIX G: CONNECTION WITH THE POST-NEWTONIAN APPROXIMATION.

Since we are working in the 3-orthogonal gauges, we can easily make contact with the "post-Newtonian approximation" of general relativity in the recent formulation of Refs. [129,130]. In this formulation one defines four potentials $V$, $V_r$, starting from the 4-metric of metric gravity: in $\Sigma_\tau$-adapted coordinates we have [we make the rescaling $\tau = c\bar{\tau}$]

$$\begin{aligned}
&\frac{1}{c^2}\,{}^4g_{\tau\tau} = {}^4\tilde{g}_{\bar{\tau}\bar{\tau}} = \frac{\epsilon}{c^2}(N^2 - {}^3g_{rs}N^rN^s) = \frac{\epsilon}{c^2}(N^2 - {}^3g^{rs}N_rN_s) = \\
&\qquad = \epsilon e^{-\frac{2}{c^2}V} = \epsilon[1 - \frac{2}{c^2}V + O(c^{-4})],\\
&\frac{1}{c}\,{}^4g_{\tau r} = {}^3\tilde{g}_{\bar{\tau}r} = -\frac{\epsilon}{c}\,{}^3g_{rs}N^s = -\frac{\epsilon}{c}N_r = \epsilon\frac{4}{c^3}V_r,\\
&{}^4g_{rs} = -\epsilon\,{}^3g_{rs} = -\epsilon e^{\frac{2}{c^2}V}\,{}^3\gamma_{rs},\\
&\qquad\Downarrow\\
&V = -\Phi = -\frac{c^2}{2}ln\,\frac{N^2/c^2}{1 + \frac{1}{c^2}\,{}^3\gamma^{rs}N_rN_s}, (Newton\ potential)\\
&V_r = -\frac{c^2}{4}N_r, (gravitomagnetic\ potential).
\end{aligned} \tag{G1}$$

The study of Einstein's equations [with the time variables rescaled by $c$] ${}^4G^{\mu\nu} \overset{\circ}{=} \frac{8\pi G}{c^4}T^{\mu\nu}$ with a matter energy-momentum tensor satisfying the post-Newtonian assumptions $T^{oo} = O(c^2)$, $T^{oi} = O(c)$, $T^{ij} = O(c^o)$, implies that, independently from the choice of a coordinate system for either $M^4$ or $\Sigma_\tau$, the 3-curvature of the auxiliary 3-metric $\gamma_{rs}$ is $O(c^{-4})$. Therefore, we can always choose 3-coordinates [algebraic spatial isotropy condition of Ref. [130]: $-{}^4g_{\bar{\tau}\bar{\tau}}\,{}^4\tilde{g}_{rs} = \delta_{rs} + O(c^{-4})$; it contains both the harmonic and the standard post-Newtonian gauges] such that

$$\begin{aligned}
&{}^3\gamma_{rs} = \delta_{rs} + O(c^{-4}),\\
&\qquad\Downarrow\\
&V = -\frac{c^2}{2}ln\,\frac{N^2/c^2}{1 + \frac{1}{c^2}\delta^{rs}N_rN_s + O(c^{-6})};
\end{aligned} \tag{G2}$$

namely, such that ${}^3\gamma_{rs}$ is a flat 3-metric at the first post-Newtonian approximation.

Since the 3-orthogonal gauges are a particular class of these 3-coordinate systems, we get the following form of the potentials $V$, $V_r$ in our rest-frame instant form of tetrad gravity without matter in the special 3-orthogonal gauge with $\rho(\tau, \vec{\sigma}) \approx 0$ [see Eqs. (60)-(65) in the Conclusions]

$$V_r(\tau, \vec{\sigma}) = -\frac{c^3}{4} \hat{n}_r(\tau, \vec{\sigma} | r_{\bar{a}}, \pi_{\bar{a}}, \phi[r_{\bar{a}}, \pi_{\bar{a}}]],$$

$$V(\tau, \vec{\sigma}) = -\Phi(\tau, \vec{\sigma}) = -\frac{c^2}{2} ln \, \frac{(-\epsilon + \hat{n})^2/c^2}{1 + \frac{1}{c^2}\delta^{rs}\hat{n}_r\hat{n}_s + O(c^{-6})} (\tau, \vec{\sigma} | r_{\bar{a}}, \pi_{\bar{a}}, \phi[r_{\bar{a}}, \pi_{\bar{a}}]],$$

$$or \;\; (from\, {}^4g_{rs})$$

$$V(\tau, \vec{\sigma}) = -\Phi(\tau, \vec{\sigma}) = c^2 \Big[ \frac{1}{\sqrt{3}} \sum_{\bar{a}} \gamma_{\bar{a}r} r_{\bar{a}} + 2ln\, \phi[r_{\bar{a}}, \pi_{\bar{a}}] \Big] (\tau, \vec{\sigma}) + O(c^{-2}). \tag{G3}$$

The two expressions of $V$ should agree at the first post-Newtonian approximation, but it is not possible to make the check in absence of the explicit knowledge of $\phi$, $\hat{n}$, $\hat{n}_r$ [moreover let us remember that both $\hat{n}$ and $\hat{n}_r$ depend on both $G/c^3$ and $c^3/G$ simultaneously and this will complicate the check].

Instead, the relation with Sections 6 and 7 of Ref. [126] [with the field $\Theta = 1$ by rescaling the absolute time] is based on the equations of that paper describing the Galileo generally covariant formulation of Newtonian gravity as a limit $c \rightarrow \infty$ on the ADM action of metric gravity. The starting point is the following parametrization of the 4-metric (we show only the 26 terms which appear in the Newtonian action)

$$\frac{1}{c^2}\, {}^4g_{\tau\tau} = {}^4\tilde{g}_{\bar{\tau}\bar{\tau}} = \epsilon [1 - \frac{2A_o}{c^2} + \frac{2\alpha_o}{c^4} + O(c^{-6})] = \epsilon e^{-\frac{2}{c^2} V},$$

$$\frac{1}{c}\, {}^4g_{\tau r} = {}^4\tilde{g}_{\bar{\tau} r} = -\epsilon [-\frac{A_r}{c} - \frac{\alpha_r}{c^3} + O(c^{-5})] = -\epsilon \Big( -\frac{4}{c^3} V_r \Big),$$

$${}^4g_{rs} = -\epsilon\, {}^3g_{rs} = -\epsilon [{}^3\check{g}_{rs} + \frac{\check{\gamma}_{rs}}{c^2} + \frac{\check{\beta}_{rs}}{c^4} + O(c^{-6})] = -\epsilon e^{\frac{2}{c^2} V}\, {}^3\gamma_{rs},$$

$$\Downarrow$$

$$\begin{aligned}
N^2 &= c^2 - 2A + \frac{2}{c^2} (\alpha_o - {}^3\check{g}^{rs} \alpha_r A_s - \frac{1}{2} \check{\gamma}_{rs}\, {}^3\check{g}^{rm}\, {}^3\check{g}^{sn} A_m A_n) + O(c^{-4}),\\
A &= A_o - \frac{1}{2}\, {}^3\check{g}^{rs} A_r A_s, \qquad {}^3\check{g}^{ru}\, {}^3\check{g}_{us} = \delta^r_s,\\
N_r &= A_r + \frac{1}{c^2} \alpha_r + O(c^{-4}).
\end{aligned} \tag{G4}$$

The final action with general Galileo covariance depends on the 26 fields $A_o$, $\alpha_o$, $A_r$, $\alpha_r$, ${}^3\check{g}_{rs}$, $\check{\gamma}_{rs}$, $\check{\beta}_{rs}$. There are 18 first class constraints and 8 pairs of second class ones. It turns out that $\alpha_o$, $A_r$, $\alpha_r$, three components of ${}^3\check{g}_{rs}$, one component of the momentum conjugate to ${}^3\check{g}_{rs}$, the trace $\check{\beta}^T$ and the longitudinal $\check{\beta}^L_r$ parts of $\check{\beta}_{rs}$ in its TT decomposition, and the longitudinal $\check{\gamma}^L_r$ part of $\check{\gamma}_{rs}$ are Hamiltonian gauge variables, while $A_o$ (the Newton potential) and the remaining components of ${}^3\check{g}_{rs}$, $\check{\gamma}_{rs}$, $\check{\beta}_{rs}$ are determined, together with their conjugate momenta, by the second class constraints. There are no propagating dynamical degrees of freedom. The gauge variables describe the inertial forces in arbitrary accelerated not-Galilean reference frames.

The post-Newtonian approximation of Refs. [129,130] implies the choices

$$A_r = 0, \qquad {}^3\check{g}_{rs} = \delta_{rs}, \qquad \check{\gamma}_{rs} = 2A_o\delta_{rs},$$

which are consistent with the previous gauge freedom (it is a possible gauge of the general Galileo covariant description of Newtonian gravity). Moreover, we have

$$V = -\Phi = A_o + \tfrac{1}{c^2}(\alpha_o - 2A_o^2) + O(c^{-4}) = A_o + O(c^{-2}),$$

$$V_r = -\tfrac{1}{4}\alpha_r + O(c^{-2}),$$

$${}^3\gamma_{rs} = \delta_{rs} + \tfrac{1}{c^4}[\check{\beta}_{rs} - 2(\alpha_o + 2A_o^2)\delta_{rs}] + O(c^{-6}) = \delta_{rs} + O(c^{-4}).$$

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